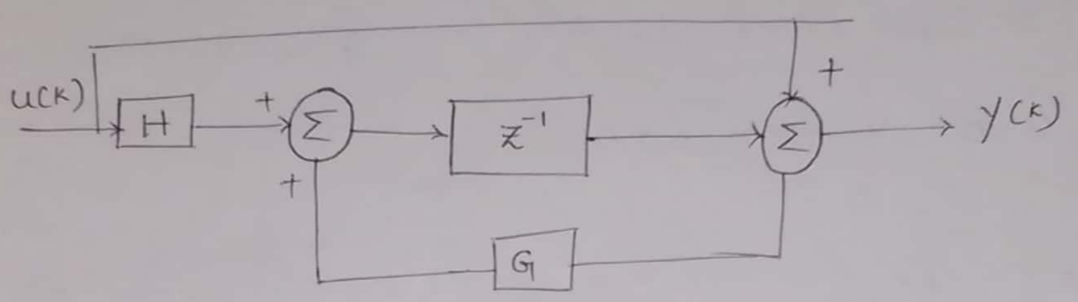
Unit - VI

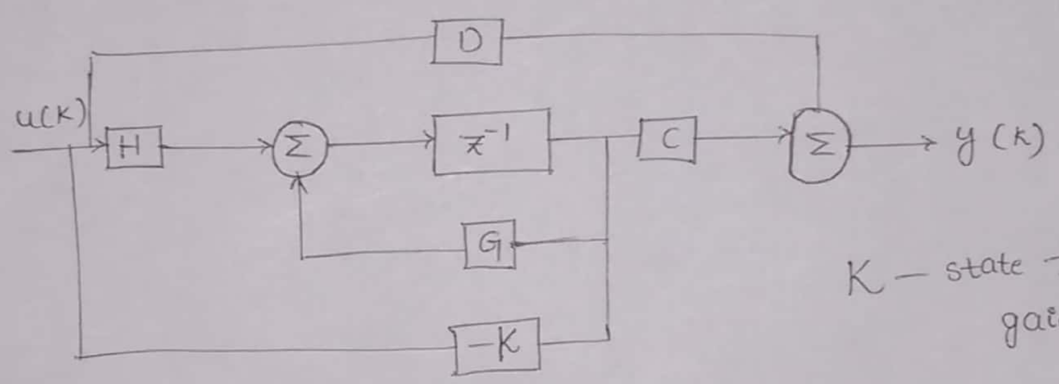
State feedback design and State Observer

State space form without feed back :-

$$\left.\begin{aligned} x(k+1) &= Gx(k) + Hu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}\right\} \quad (1)$$



with state feed back :-



K - state feed back gain matrix

$$x(k+1) = Gx(k) + H(-k\, x(k))$$

$$= (G - HK)\, x(k) \quad \text{—— 2}$$

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Design statement :-

Design a state feed back gain matrix (K) such that, the control law is given by

$u(k) = -Kx(k)$ places the poles of the closed loop system with state feed back given by equation 2 in desired location.

Design of K (gain matrix) by transforming state model into controllable canonical form :-

Step-1(i): If $x(k+1) = Gx(k) + Hu(k)$ —— (1) not in controllable canonical form (CCF)

(ii): Define a matrix
$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 & 1 \\ a_{n-2} & a_{n-3} & & a_1 & 1 & 0 \\ \vdots & & & & & \\ a_1 & 1 & & 0 & 0 & 0 \\ 1 & 0 & & 0 & 0 & 0 \end{bmatrix}$$

Step-2 :

If C_N is controllability matrix is given by

$$C_N = \left[H \,\vdots\, GH \,\vdots\, G^2H \,\vdots\, \cdots\cdots \, G^{n-1}H \right]$$

Step-3 :

Define new state variables by $x(k)$

$$x(k) = T\bar{x}(k)$$

$x(k)$ ↓ old state variable matrix

$\bar{x}(k)$ ↓ New state variable matrix.

Then

$$x(k) = T\bar{x}(k)$$

$$1 \Rightarrow \quad T\bar{x}(k+1) = GT\bar{x}(k) + Hu(k)$$

$$\bar{x}(k+1) = T^{-1}GT\bar{x}(k) + T^{-1}Hu(k)$$

$$\bar{x}(k+1) = \bar{G}\bar{x}(k) + \bar{H}u(k) \quad \text{--- } (2)$$ is in controllable canonical form

where. $$\boxed{T = C_M W} \quad \text{--- } (3)$$

If the poles are to be placed at $P_1, P_2 \ldots P_n$ then the desired characterstic equation is

$(z-P_1)(z-P_2)\ldots\ldots(z-P_n)$ it is required C.E after simplyfing the above equation,

Let the desired C.E will be in the form of

$$z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \ldots\ldots + \alpha_{n-1} z + \alpha_n = 0 \quad \text{--- } (4)$$

→ If a state feed back is introduced the new system matrix will become

$$\bar{x}(k+1) = \bar{G}\bar{x}(k) - \bar{K}\bar{H}\bar{x}(k) \leftarrow$$ [In eq. 2 sub $u(k) = -\bar{K}\bar{x}(k)$]

$$= (\bar{G} - \bar{K}\bar{H})\bar{x}(k)$$

$$\bar{G} - \bar{H}\bar{K} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \bar{k}_1 & \bar{k}_2 & \cdots & k_n \end{bmatrix}$$

$$\overline{G}' = \overline{G} - \overline{H}\,\overline{K}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ -(a_n+\overline{K}_1) & -(a_{n-1}+\overline{K}_2) & \cdots & \cdots & -(a_1+\overline{K}_n) \end{bmatrix}$$

From above matrix writing in a polynomial form we have

$$\Rightarrow z^n + (a_1+\overline{K}_n) z^{n-1} + (a_2+\overline{K}_{n-1}) z^{n-2} + \cdots + (a_{n-1}+\overline{K}_{n-1}) z + (a_n+\overline{K}_1) = 0 \quad \cdots\cdots (5)$$

we get $\overline{K}$ values by comparing equations of 4 & 5

$$\overline{K} = [\alpha_n - a_n \quad \alpha_{n-1} - a_{n-1} \quad \alpha_{n-2} - a_{n-2} \cdots \quad \alpha_2 - a_2 \quad \alpha_1 - a_1]$$

The required state-feed back gain matrix $\boxed{K = \overline{K}\, T^{-1}}$

PRO

Design k for the following system such that closed loop poles are located at 0.5, 0.6, 0.7.

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

Given system is controllable.

solution $\quad z^3 + 3z^2 + 2z + 1 = 0$

$$a_1 = 3 \; ; \; a_2 = 2 \; ; \; a_3 = 1$$

required pole location $0.5, 0.6, 0.7$

C.E of system with feedback $(z-0.5)(z-0.6)(z-0.7)=0$

$\Rightarrow (z-0.5)(z^2-0.6z-0.7z+0.42)=0$

$\Rightarrow (z-0.5)(z^2-1.3z+0.42)=0$

$\Rightarrow z^3-0.5z^2-1.3z^2+0.65z+0.42z-0.21=0$

$\Rightarrow z^3-1.8z^2+1.07z-0.21=0$

$\alpha_1 = -1.8 \; ; \; \alpha_2 = 1.07 \; ; \; \alpha_3 = -0.21$

$K = [K_1 \quad K_2 \quad K_3] = [\alpha_3 - a_3 \quad \alpha_2 - a_2 \quad \alpha_1 - a_1]$

$= [-1.21 \quad -0.93 \quad -4.8]$

Pro

Consider the system described by

$$x(k+1) = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

obtain the state feedback matrix K to place the poles 0.5 & 0.6, if the system is state controllable.

solution :-

$T = C_N W$

$C_N = [H \,\vdots\, GH \,\vdots\, \cdots \, G^{n-1}H]$

$= [H \,\vdots\, GH]$ $\therefore$ $GH = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

W is from C.E of given system.

$$|zI - G| = 0$$

$$\left| \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \right| = 0 \quad ; \quad W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & & & & \\ 1 & \cdots & & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} z+1 & 1 \\ 0 & z+2 \end{vmatrix} = 0 \Rightarrow z^2 + 3z + 2 = 0 \Leftrightarrow (z+1)(z+2)$$

$$a_1 = 3 \; ; \; a_2 = 2$$

a_{n-1}

$$W = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T = C_H W$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\frac{1}{ad-bc} \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$\bar{G} = T^{-1} G T = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$z^2 + 3z + 2 = 0$$

X

required pole location

$$(z-0.5)(z-0.6)=0$$

$$\Rightarrow z^2-1.1z+0.3 \quad ; \quad \alpha_1=-1.1 \quad ; \quad \alpha_2=0.3$$

$$\bar{K}=[\bar{K}_1, \bar{K}_2]=[\alpha_2-a_2 \quad \alpha_1-a_1]$$

$$=[-1.7 \quad -4.1]T^{-1}$$

Pro. Design K to place the poles at 0.5, 0.4 ± j0.4

$$x(k+1)=Gx(k)+Hu(k)$$

$$G=\begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix} \quad ; \quad H=\begin{bmatrix} 0.01 \\ 0 \\ 0.005 \end{bmatrix}$$

$(z-0.4-j0.4)$
$(z-0.4+j0.4)$

$W = a_1, a_2$

SOL :-

$$|zI-G|=0$$

I ; $|zI-G|=0$

$$\begin{vmatrix} z-0.1 & 0 & -0.1 \\ 0 & z-0.5 & -0.2 \\ -0.2 & 0 & z-0.4 \end{vmatrix}=0$$

$$\Rightarrow (z-0.1)\{(z-0.5)(z-0.4)-0\}-0.1(0.2(z-0.5))=0$$

$$\Rightarrow (z-0.1)[z^2-0.5z-0.4z+0.2]-0.02(z-0.5)=0$$

$$\Rightarrow z^3-0.1z^2-0.9z+0.09z+0.2z-0.02z-0.02+0.01=0$$

$$\Rightarrow z^3-z^2+0.27z-0.01=0$$

$a_3 = -0.01 \; ; \; a_2 = 0.27 \; , \; a_1 = -1.$ $\quad W = \begin{bmatrix} 0.27 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Required C.E @ required poles

$\Rightarrow (z-0.5)(z-0.4+j0.4)(z-0.4-j0.4) = 0$

$\Rightarrow (z-0.5)(z^2 - 0.4z + \cancel{j0.4z} - 0.4z + 0.16 - \cancel{j0.16} - \cancel{j0.4z}$

$+ \cancel{j0.16} + 0.16) = 0$

$\Rightarrow (z-0.5)(z^2 - 0.8z + 0.32) = 0$ pole @ $j\omega$.

$\Rightarrow z^3 - 0.5z^2 - 0.8z^2 + 0.4z + 0.32z - 0.16 = 0$

$\Rightarrow z^3 - 1.3z^2 + 0.72z - 0.16 = 0$

$\alpha_3 = -0.16 \quad \alpha_2 = 0.72z \; , \quad \alpha_1 = -1.3$

$\bar{K} = [\alpha_3 - a_3 \quad \alpha_2 - a_2 \quad \alpha_1 - a_1]$

$= [-0.15 \quad 0.43 \quad -0.3]$

$K = \bar{K}\, T^{-1}$

$T = C_N W$

$C_N = [H \mid GH \mid G^2H]$

$= \begin{bmatrix} 0.01 & 1.5\times10^{-3} & 5.5\times10^{-4} \\ 0 & 1\times10^{-3} & 1.3\times10^{-3} \\ 0.005 & 4\times10^{-3} & 1.9\times10^{-3} \end{bmatrix}$

$T = \begin{bmatrix} 0.01 & 1.5\times10^{-3} & 5.5\times10^{-4} \\ 0 & 1\times10^{-3} & 1.3\times10^{-3} \\ 0.005 & 4\times10^{-3} & 1.9\times10^{-3} \end{bmatrix} \begin{bmatrix} 0.27 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\Rightarrow T^{-1}$

$$T = \begin{bmatrix} 1.75\times10^{-3} & -8.5\times10^{-3} & 0.01 \\ 3\times10^{-4} & 1\times10^{-3} & 0 \\ -7.5\times10^{-4} & -1\times10^{-3} & 5\times10^{-3} \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 192.3 & 1250 & -384.6 \\ -57.69 & 625 & 115.38 \\ 17.3 & 312.5 & 165.38 \end{bmatrix}$$

$$K = \bar{K} T^{-1} = \begin{bmatrix} -0.15 & 0.43 & -0.3 \end{bmatrix} \begin{bmatrix} 192.3 & 1250 & -384.6 \\ -57.69 & 625 & 115.38 \\ 17.3 & 312.5 & 165.38 \end{bmatrix}$$

$$= \begin{bmatrix} -58.84 & -12.5 & 57.69 \end{bmatrix}$$

pro

$$x[(k+1)T] = Ax(kT) + Bu(kT)$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

obtain state feed back gain matrix K. such that the closed loop poles are at 0.1, 0.2.

SOL :— From the given poles required C.E

$$(z-0.1)(z-0.2) = z^2 - 0.3z + 0.02$$

$$\alpha_1 = -0.3, \quad \alpha_2 = 0.02$$

=> system is in controllable canonical form

$$a_1 = 2 \qquad a_2 = 1$$

$$K = \begin{bmatrix} \alpha_2 - a_2 & \alpha_1 - a_1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.98 & -2.3 \end{bmatrix}$$

Prob From the above given data. Eg $A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution :-

$$C_N = \begin{bmatrix} B & \vdots & AB \end{bmatrix}. \qquad \because AB = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$$

$$|zI - A| = 0$$

$$\Rightarrow \left| \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \right| = \left| \begin{bmatrix} z-1 & -1 \\ 1 & z+2 \end{bmatrix} \right| = 0$$

$$\Rightarrow (z-1)(z+2) + 1 = 0$$

$$a_1 = 1, \quad a_2 = -1.$$

$$W = \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T = C_N W = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 0.2 & -0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

$$K = [\alpha_2 - a_2 \quad \alpha_1 - a_1] T^{-1}$$

$$= [1.02 \quad -1.3] \begin{bmatrix} 0.2 & -0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= [-0.316 \quad -0.984]$$

Ackerman's formula :-

It gives a closed loop form of expression from which the state feed back gain matrix "K" is directly calculated.

whose

$$x(k+1) = Gx(k) + Hu(k) \quad \text{—} (1)$$ single i/p without feed back.

where

$x(k)_{n\times 1}$ vector, $[G]$ = $n\times n$ matrix, $H = n\times 1$ vector $u(k)$ = scalar

Now if a feed back is given by

$$u(k) = -K_{1\times n}\, x(k)_{n\times 1} \quad (2)$$

The system of equation (1) will become

$$x(k+1) = Gx(k) - HK\,x(k)$$

$$x(k+1) = (G - HK)\,x(k) \quad \text{—} 2$$ single i/p with feed back

Let the desired poles are $P_1, P_2 \dots P_n$

The C.E of the system with feedback

$$\Rightarrow |zI - (G - HK)| = 0$$

$$\Rightarrow (z-P_1)(z-P_2)\dots(z-P_n) = |zI - (G-HK)| = 0$$

$\Rightarrow z^n + \alpha_1 z^{n-1} + \cdots \alpha_{n-1} z + \alpha_n = 0$ is of $G-H = \overline{G}$

according to calay - Hamoittion theorem

$$\phi(\overline{G}) = \overline{G}^n + \alpha_1 \overline{G}^{n-1} + \cdots\cdots + \alpha_{n-1}\overline{G} + \alpha_n [I] = 0 \quad (3)$$

$\alpha_n I = I$

$\alpha_{n-1} \quad \overline{G} = G - HK$ $\qquad K = [\quad]_{1\times n} C_N^{-1} \phi(\overline{G})$

$\alpha_{n-2} \overline{G}^2 = (G-HK)(G-HK)$

$= G^2 - GHK - HK\overline{G}$

$\overline{G}^3 = (G-HK)^2 (G-HK)$

$= (G^2 - GHK - HK\overline{G})(G-HK)$

$= G^3 - G^2HK - GHK\overline{G} - HK\overline{G}^2$ $\qquad \overline{G} = G - HK$

$\alpha_0 \times \overline{G}^n = G^n \cdots\cdots - HK\overline{G}^{n-1}$

adding the above equations :-

$$\Rightarrow \underbrace{\alpha_n I + \alpha_{n-1}\overline{G} + \alpha_{n-2}\overline{G}^2 + \cdots + \alpha_1 \overline{G}^{n-1} + G}_{\phi(\overline{G})} = \alpha_n I + \alpha_{n-1} G - \alpha_{n-1} HK + \alpha_{n-2} G^2 - \alpha_{n-2} GHK - \alpha_{n-2} HK\overline{G} + \cdots + \alpha_0 G^n + \cdots - \alpha_0 HK\overline{G}^n$$

From eq (3) L.H.S of above equation is becomes.

$$\Rightarrow \phi(\overline{G}) = \alpha_n I + \alpha_{n-1} G + \alpha_{n-2} G^2 + \cdots G^n - [H \mid GH \mid \cdots G^{n-1}H] \begin{bmatrix} \alpha_{n-1}K + \alpha_{n-2}K\overline{G} + \cdots + K\overline{G}^{n-1} - (\phi_1) \\ \alpha_{n-2} K \text{ (assume } \phi_2) \\ K \end{bmatrix}$$

$$\Rightarrow \phi(\bar{G}) = \phi(G) - C_N \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ K \end{bmatrix}_{n\times 1}$$

from eq (3) $\phi(\bar{G}) = 0$

$$\Rightarrow 0 = \phi(G) - C_N \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ K \end{bmatrix} \Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ K \end{bmatrix}_{n\times 1} = C_N^{-1} \phi(G)$$

premultiply with $[0 \; 0 \; 0 \; \cdots \; 1]$

$$\boxed{K = [0 \; 0 \; \cdots \; 1]_{1\times n} \, C_N^{-1} \, \phi(G)}$$

pro. Using Ackerman's formula determine the matrix "K".

$$G = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix} \quad H = \begin{bmatrix} 0.01 \\ 0 \\ 0.05 \end{bmatrix}$$ poles @ 0.5, $0.4 \pm j0.4$

Sol:—

$$K = [0 \; 0 \; 1] \, C_N^{-1} \, \phi(G)$$

$$\phi(G) = G^3 + \alpha_1 G^2 + \alpha_2 G + \alpha_3 I = 0$$

$(z-0.5)$

$(z-0.4-j0.4)$

from the poles required C.E is

$$(z-0.5)(z-0.4+j0.4)(z-0.4-j0.4)$$

$$\Rightarrow z^3 - 1.3z^2 + 0.72z - 0.16 = 0$$

$$\alpha_1 = -1.3, \quad \alpha_2 = 0.72, \quad \alpha_3 = -0.16$$

$$G^2 = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix} \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.03 & 0 & 0.05 \\ 0.04 & 0.25 & 0.18 \\ 0.1 & 0 & 0.18 \end{bmatrix}$$

$$G^3 = \begin{bmatrix} 0.013 & 0 & 0.023 \\ 0.04 & 0.125 & 0.126 \\ 0.046 & 0 & 0.082 \end{bmatrix}$$

$$\phi(G) = \begin{bmatrix} 0.013 & 0 & 0.023 \\ 0.04 & 0.125 & 0.126 \\ 0.046 & 0 & 0.082 \end{bmatrix} - 1.3 \begin{bmatrix} 0.03 & 0 & 0.05 \\ 0.04 & 0.25 & 0.18 \\ 0.1 & 0 & 0.18 \end{bmatrix}$$

$$+ 0.72 \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix} - 0.16 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\phi(G) = \begin{bmatrix} -0.114 & 0 & 0.03 \\ -0.012 & 0 & 0.036 \\ 0.06 & 0 & -0.024 \end{bmatrix}$$

$\Rightarrow$ CN

$$GH = \begin{bmatrix} 0.1 & 0 & 0.01 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0 \\ 0.005 \end{bmatrix} = \begin{bmatrix} 0.0015 \\ 0.001 \\ 0.004 \end{bmatrix}$$

$$G^2 H = \begin{bmatrix} 0.00055 \\ 0.0013 \\ 0.0019 \end{bmatrix} = \begin{bmatrix} 5.5\times10^{-4} \\ 1.3\times10^{-3} \\ 1.9\times10^{-3} \end{bmatrix}$$

$$C_N = \begin{bmatrix} 0.01 & 0.0015 & 0.00055 \\ 0 & 0.001 & 0.0013 \\ 0.005 & 0.004 & 0.0019 \end{bmatrix}$$

$$C_N^{-1} = \begin{bmatrix} 89.18 & -17.56 & 37.83 \\ 1756.7 & -229.7 & -351.35 \\ -1351.3 & 945.9 & 270.2 \end{bmatrix}$$

CN^{-1}

$$K = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 126.92 & 25 & -53.8 \\ -250 & -625 & 500 \\ 192.3 & 1250 & -384.6 \end{bmatrix} \begin{bmatrix} -0.114 & 0 & 0.03 \\ -0.012 & 0 & 0.036 \\ 0.06 & 0 & -0.024 \end{bmatrix}$$

$$K = \begin{bmatrix} -60 & 0 & 60 \end{bmatrix}$$

pro

$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ obtain K to place the closed loop poles @ 0.1, 0.2

Sol:— C.E from the poles

$\Rightarrow (z-0.1)(z-0.2) = 0$

$\Rightarrow z^2 - 0.3z + 0.02 = 0$

$$\alpha_1 = -0.3 \, , \quad \alpha_2 = 0.02$$

$$\phi(G) = G^2 + \alpha_1 G + \alpha_2 I$$

$$G^2 = A^2 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$$

$$\phi(G) = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix} + (-0.3) \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix}$$

$$= \begin{bmatrix} -0.98 & -2.3 \\ 2.3 & 3.62 \end{bmatrix}$$

$$C_N = \left[H \,\vdots\, GH \right] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$C_N^{-1} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -0.98 & -2.3 \\ 2.3 & 3.62 \end{bmatrix}$$

$$= \begin{bmatrix} -0.98 & -2.3 \end{bmatrix}$$

If the system is defined as

$$x(k+1) = Gx(k) + Hu(k)$$

The controllability matrix C_N is C_H – controllability matrix.

$$C_N = \left[H \;\vdots\; GH \;\vdots\; \cdots \; G^{n-1}H \right]$$

where n is the size of the G matrix or order of the system

→ If the system is controllable and state feed back design can be done by the following way

1 a If the system is already in controllable form

1 b) If the system is not in controllable form

2. Ackerman's formula

1 a :

$$G = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

step 1 :— From the G matrix obtain $a_1, a_2 \cdots a_n$ (i.e.,

step 2 :— If the desired poles given in the problem are $P_1, P_2, P_3 \cdots P_n$, obtain $\alpha_1, \alpha_2 \cdots \alpha_n$ as below

$$(z-P_1)(z-P_2)\cdots(z-P_n) = 0 \text{ on multiplying we get}$$

$$\Rightarrow z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \cdots + \alpha_{n-1} z + \alpha_n = 0.$$

step 3 :— The state feed back gain matrix,

$$K = \left[\alpha_n - a_n \quad \alpha_{n-1} - a_{n-1} \quad \alpha_1 - a_1\right]_{1\times n}$$

1 b)

step 1 : obtain $a_1, a_2 \cdots a_n$ from $|zI - G| = 0$ which gives

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} \cdots + a_{n-1} z + a_n = 0$$

step 2 :

If the desired poles are $P_1, P_2 \cdots P_n$

$$(z - P_1)(z - P_2) \cdots (z - P_n) = 0$$

$$z^n + \alpha_1 z^{n-1} + \cdots \alpha_{n-1} z + \alpha_n = 0$$

obtain $\alpha_1, \alpha_2 \cdots \alpha_n$

step 3 :

$$C_N = \left[H \,\vdots\, GH \,\vdots\, \cdots \; G^{n-1} H \right]$$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & & a_2 & a_1 & 1 \\ a_{n-2} & a_{n-3} & & a_1 & 1 & 0 \\ \vdots & & & & & \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

step 4 : $T = C_N W \Rightarrow$ find T^{-1}

step 5 : The state feed back gain matrix

$$K = \underbrace{\left[\alpha_n - a_n \quad \alpha_{n-1} - a_{n-1} \quad \alpha_1 - a_1 \right]}_{K} T^{-1}.$$

⑩

2. Ackerman's formula [for both controllable form & Non-controllable form] :-

→ If desired poles are $p_1, p_2 \ldots p_n$

$$(z-p_1)(z-p_2)\ldots(z-p_n)=0$$

$$\Rightarrow z^n + \alpha_1 z^{n-1} + \ldots + \alpha_{n-1} z + \alpha_n = 0$$

Then $\phi(G) = G^n + \alpha_1 G^{n-1} + \ldots + \alpha_{n-1} G + \alpha_n I = 0$

$$C_M = \left[H \mid GH \mid \ldots \mid G^{n-1}H\right]$$

⇒ findout C_M^{-1}

⇒ state feed back (K)

$$\boxed{K = [0\ 0 \ldots 1]_{1\times n}\ C_M^{-1}\ \phi(G)}$$

Proof of necessary condition for pole placement by state feed back (K) :-

The rank of matrix is the no. of independent colomn.

let us design matrix, such that it is having

$$P = \left[f_1 \mid f_2 \mid \ldots f_q \mid v_{q+1} \mid v_{q+2} \mid \ldots v_n\right]$$

let us assume the rank of C_M matrix $(q < n)$

q - independent column

n-q - dependent columns

Define new state variable

$$x(k) = P\bar{x}(k)$$

$$x(k+1) = G\,x(k) + H\,u(k) \;; \quad u(k) = -K\,x(k) \quad -1$$

$$P\bar{x}(k+1) = GP\bar{x}(k) + H\,U(k)$$

$$\bar{x}(k+1) = P^{-1}GP\bar{x}(k) + P^{-1}HU(k)$$

$$\bar{x}(k+1) = P^{-1}GP\bar{x}(k) - P^{-1}HKx(k)$$

$$\bar{x}(k+1) = (\bar{G} - \bar{H}K)\bar{x}(k) \qquad \text{where} \quad \bar{G} = P^{-1}GP$$

$$\bar{G} = P^{-1}GP = \left[\begin{array}{cccc|cccc} g_{11} & g_{12} & \cdots & g_{1q} & g_{1q+1} & g_{1q+2} & \cdots & g_{1n} \\ g_{q1} & g_{q2} & \cdots & g_{qq} & g_{qq+1} & g_{qq+2} & \cdots & g_{qn} \\ \hline & & & & g_{q+1\,q+1} & \cdots & \cdots & g_{q+1\,n} \\ & & 0 & & & & & \\ & & & & g_{nq+1} & \cdots & \cdots & g_{nn} \end{array}\right]$$

$$\bar{G} = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] \;; \quad [G_{11}]_{q\times q} \;; \quad G_{12} = [\;\;]_{q\times n-q}$$

$$G_{21} = [0]_{n-q\times q} \qquad G_{22} = [\;\;]_{n-q\times n-q}$$

$$\bar{H} = P^{-1}H = \begin{bmatrix} h_{11} \\ h_{21} \\ \vdots \\ h_{q1} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} q \text{ rows} \\ \\ n-q \text{ rows} \end{array} \qquad \bar{H} = \begin{bmatrix} H_{11} \\ \hline 0 \end{bmatrix}$$

$$\bar{K} = \left[K_{11} \mid K_{12} \right]_{1\times n} \; ; \; K_{11} = [\;]_{1\times q} \; ; \; K_{12} = [\;]_{1\times n-q}$$

Once the feedback is given means the desired poles are alrea placed i.e., the roots of characterstic equation of the system defined by the equation 2. is nothing but the desired pole

$$|z\bar{I} - \bar{G} + \bar{H}\bar{K}| = 0$$

$$\bar{I} = \begin{array}{c} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & & & 0 \\ 0 & 1 & 0 & \cdots & & & 0 \\ & & & & q\times q & & q\times n-q \\ \hline \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & & & 1 \end{array}\right] \begin{array}{l} \}\, q \text{ rows} \\ \\ \}\, n-q \text{ rows} \end{array} \\ \quad\quad n-q\times q \quad\quad n-q\times n-q \end{array}$$

$$\bar{I} = \begin{bmatrix} I_{q\times q} & 0 \\ 0 & I_{n-q\times n-q} \end{bmatrix}$$

$$\Rightarrow \left| \begin{bmatrix} zI_q & 0 \\ 0 & zI_{n-q} \end{bmatrix} - \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix} + \begin{bmatrix} H_{11} \\ 0 \end{bmatrix} \left[K_{11} \mid K_{12} \right] \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} zI_q & 0 \\ 0 & zI_{n-q} \end{bmatrix} - \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix} + \begin{bmatrix} H_{11}K_{11} & H_{11}K_{12} \\ 0 & 0 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} (zI_q - G_{11} + H_{11}K_{11})_{q\times q} & (-G_{12} + H_{11}K_{12})_{q\times n-q} \\ 0 & zI_{n-q} - G_{22} \end{vmatrix} = 0$$

$$\Rightarrow \left(zI_q - G_{11} + H_{11}K_{11}\right)\left(zI_{n-q} - G_{22}\right) = 0$$

(11)

Observing above eqn K_{12} has no control over corresponding $n-q$ eigen values. Hence the necessary condition is proved

1, 2, 7, 26, 31, 35, 40, 60, 61, 62,
88, 95, B4, E9, F6,
G1, G3, G8, H3.

L - 2, 8, 10, 12, 17, 18,
26, 43.