UNIT-6

63

6.1 a

STABILITY ANALYSIS In z-plane.

Mapping between s-plane & z-plane :-

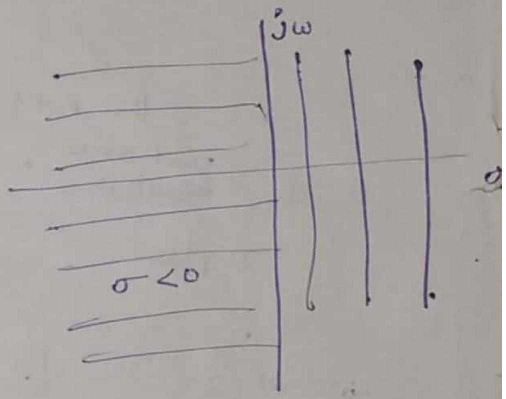
→ Since the complex variables z and s are related by $z = e^{Ts}$, the pole and zero locations in the z-plane are related to the pole and zero locations in the s-plane.

let $s = \sigma + j\omega$.

then $z = e^{Ts} = e^{T\sigma} e^{jT\omega}$.

$= e^{T\sigma} e^{j(T\omega + 2\pi k)}$

i.e There are infinitely many values of s for each value of z



⇒ splane

$s = \sigma + j\omega$.

if σ is negative → left half splane

if σ is 0 → $j\omega$ axis.

if σ is positive → Right half side

→ $z = e^{Ts} = e^{T\sigma} e^{j(T\omega + 2\pi k)}$.

$|z| = e^{T\sigma}$, $\angle z = \omega T$

let $z = u + jv$.

$|z| = u^2 + v^2$

~~if $\sigma < 0$ (left half splane) → $|z|$~~

from (1) ⇒ if $\sigma = 0$ ($j\omega$ axis) ⇒ $|z| = 1$, ← eqn of circle.

i.e $j\omega$ axis in s-plane maps to unit circle in z-plane

10, 14, 24, 26, 35, 43, 63, 64, 71, 88, 91, G, P2, F2, F3, 48, H3, L-6, 12, 15, 17, 23, 29, 34

if $\sigma < 0$ $\quad |z| = e^{-T\sigma} < 1.$
(L.H.S)

i.e circle whose radius is less than 1.
or interior of unit circle.

if $\sigma > 0$ $\quad |z| = e^{T\sigma} > 1$ outside of unit circle.

→ angle of z $\angle z = \omega T$

let ω_s = sampling frequency = $\frac{2\pi}{T}$.

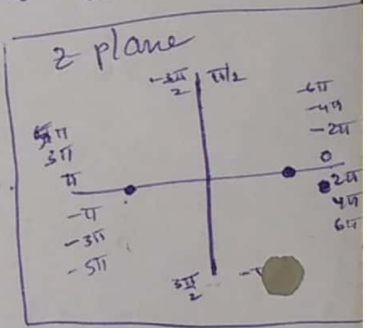
∴ angle varies from $-\infty$ to ∞ as ω varies from ∞ to $-\infty$

(1) if a point moves from $-j\frac{1}{2}\omega_s$ to $j\frac{1}{2}\omega_s$ in splane.

then if $\omega = \frac{1}{2}\omega_s \Rightarrow \angle z = \frac{1}{2} \cdot \frac{2\pi}{T} \cdot T = \pi$

if $\omega = -\frac{1}{2}\omega_s \Rightarrow \angle z = -\frac{1}{2} \cdot \frac{2\pi}{T} \cdot T = -\pi$

i.e the point from $-\frac{1}{2}j\omega_s$ to $\frac{1}{2}j\omega_s$ in s-plane maps to a ~~Semicircle~~ unit circle starting from $-\pi$ to π in C.C.W direction in z-plane.



→ (2) if $\omega = \frac{1}{2}\omega_s \Rightarrow \angle z = \pi$

$\omega = \frac{3}{2}\omega_s \Rightarrow \angle z = 3\pi$

i.e a point on $j\omega$ axis of S-plane which starts from $j\frac{1}{2}\omega_s$ to $j\frac{3}{2}\omega_s$ will map to a unit circle in the z plane in C.C.W direction.

→ $j\omega$ axis in the of splane → unit circle in zplane

maps to a unit circle.

each strip of width ω_s in L.H.S →

→ The impact of sampling on a continuous signal is reflected defining of primary strip and complementary strip.

frequency boundarys of primary strip $\pm \frac{w_s}{2}$

frequency boundaries of complementary strips = multiples of $\frac{w_s}{2}$.

This phenomenon is caused by the periodic nature of z as a function of $s = \sigma + j\omega$.

$$z = e^{Ts} = e^{T(\sigma + j\omega)} = e^{T\left(\sigma + j\omega \pm \frac{2\pi K}{T}\right)}.$$

where $w_s \to$ sampling frequency

$T \to$ sampling time

$K = 0, 1, 2 \ldots$

i.e. poles and zeros of the s-plane transfer function outside the primary strip are "folded" into the primary strip, possibly causing undesired performance.

→ The folded poles and zeros act as if they were originally in the primary strip, thus changing the system performance.

→ The s-plane poles $s = j\omega$ and $s = -j\omega$ in the primary strip are mapped to the z-plane poles $z = e^{j\omega T}$ & $z = e^{-j\omega T}$.

But these z-plane poles are also maps of s-plane poles

$s = j\omega \pm \frac{2\pi}{T},\ j\omega \pm \frac{4\pi}{T},\ j\omega \pm \frac{6\pi}{T} \ldots$ &

$s = -j\omega \pm \frac{2\pi}{T}\ \ -j\omega \pm \frac{4\pi}{T},\ -j\omega \pm \frac{6\pi}{T} \ldots$ in the complementary strip. This means there are infinitely many values of s for each z.

→ Thus the largest frequency we can distinguish is $\omega = \frac{\pi}{T}$ ~~(or)~~ (or) $\frac{w_s}{2}$, which is half of the sampling frequency.

→ Consider a representative point on $j\omega$ axis in s-plane.

As this point moves from $-j\frac{1}{2}\omega_s$ to $j\frac{1}{2}\omega_s$

$\angle z \xrightarrow[\text{varies}]{} -\pi$ to π in C.W.

As this point moves from $j\frac{1}{2}\omega_s$ to $j\frac{3}{2}\omega_s$.

the corresponding point traces out the unit circle once in the counter clockwise direction.

→ Thus as the point moves from $-\infty$ to ∞ on $j\omega$ axis, we trace the unit circle in the z plane an infinite no. of times.

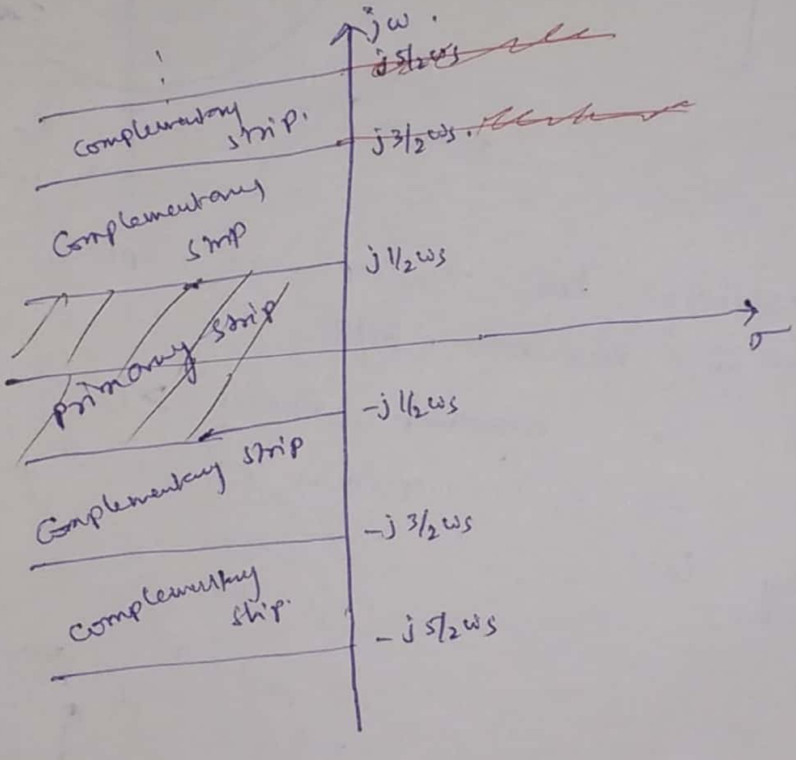
i.e each strip of width ω_s in the left half of the s plane maps into the inside of unit circle in the z-plane.

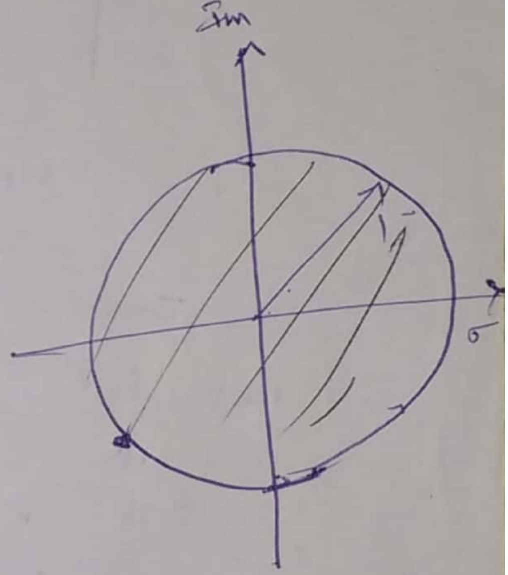
→ i.e we may divide the left half of s-plane into an infinite number of periodic strips of width ω_s.

$-j\frac{1}{2}\omega_s$ to $j\frac{1}{2}\omega_s$ ⇒ Primary strip (maps to a circle)

$j\,1/2\,\omega_s$ to $j\,3/2\,\omega_s$
$j\,3/2\,\omega_s$ to $j\,5/2\,\omega_s$
⋮
$-j\,1/2\,\omega_s$ to $-j\,3/2\,\omega_s$
$-j\,3/2\,\omega_s$ to $-j\,5/2\,\omega_s$
⋮

$\Bigg\}$ ⇒ Complementary strip. (maps on the same circle).





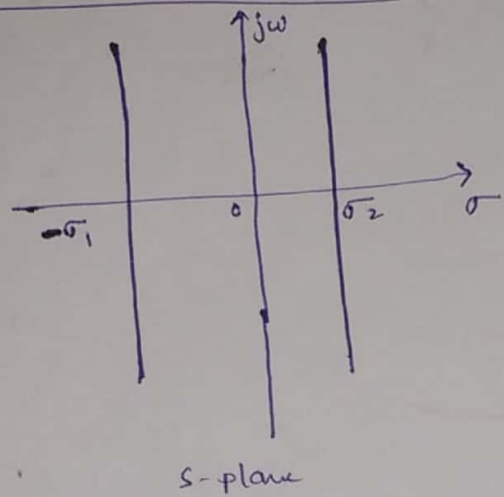
Conclusion

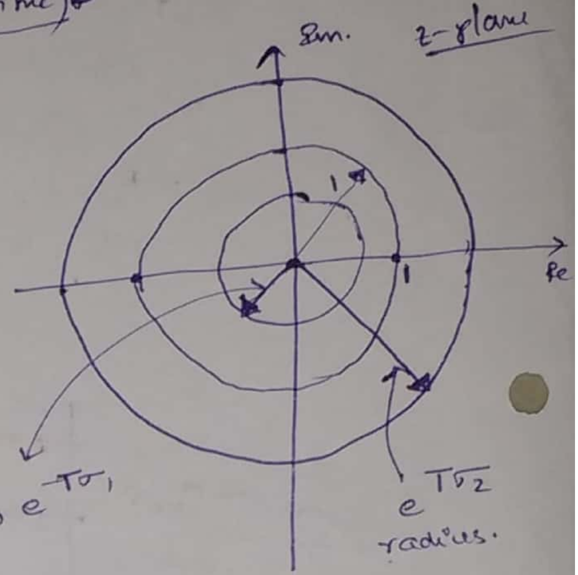
1. The correspondance between z-plane and s-plane is not unique.
2. A point in the z-plane corresponds to an infinite number of points in the splane.
3. But, A point in the s splane corresponds to a single point in the z-plane.

⇒ Mapping of some common contours in s-plane to z-plane :-

1. Constant attenuation loci i.e σ = Constant loci
2. Constant frequency loci i.e ω = Constant loci
3. Constant damping loci i.e

1. Constant attenuation loci :- (settling time)



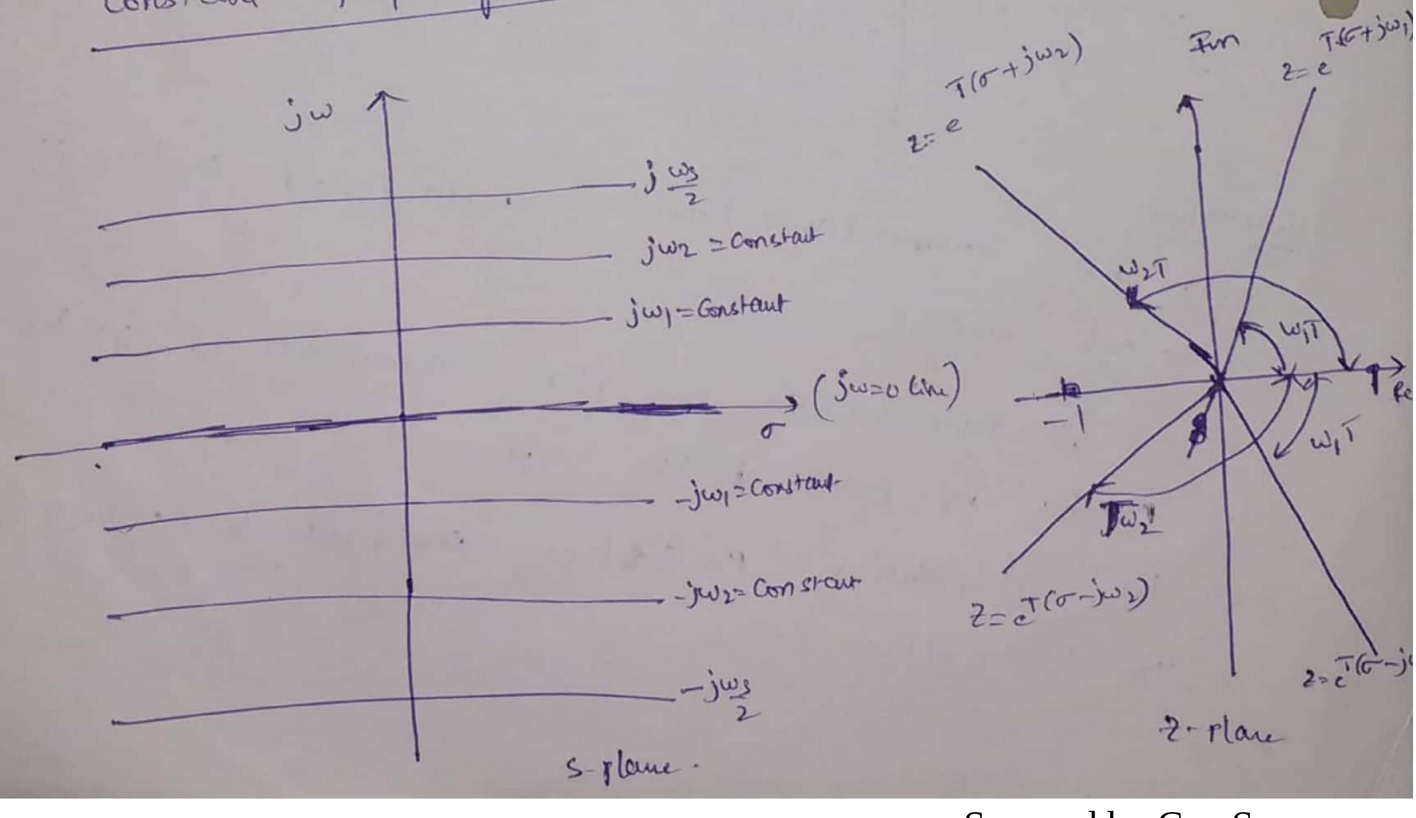


i.e Constant attenuation ~~loci~~ line in the s plane maps to a circle of radius $e^{T\sigma}$.

if $\sigma > 0$ ⇒ ~~exterior~~ radius >1

if $\sigma < 0$ ⇒ radius <1.

2. Constant frequency loci :-



$\omega = \pm \frac{1}{2}\omega_s$ line in splane in Left Half $\xrightarrow{\text{maps}}$ negative real axis in the z-plane between 0 & -1

$\omega = \pm \frac{1}{2}\omega_s$ line R. H. splane $\xrightarrow{\text{maps}}$ negative real axis of in z-plane between -1 to -∞.

$\omega = \omega_1$ (a constant) line in S-plane $\xrightarrow{\text{maps}}$ a radial line of constant angle $T\omega_1$ (radians). in z-plane.

negative real axis of s-plane $\xrightarrow{\text{maps}}$ Positive real axis of z-plane between 0 and 1.

$\omega = \pm n\omega_s$ $(n = 0, 1, 2 \dots)$ in R.H. splane $\longrightarrow$ Positive real axis in z-plane between 1 and ∞.

(3) <u>Constant - Damping Ratio - Locis -</u>

$s = -\xi\omega_n + j\omega_d$ is the Constant - Damping Ratio line.

$\omega_d = \omega_n\sqrt{1-\xi^2}$

$T = \frac{2\pi}{\omega_s}$

$$z = e^{Ts} = e^{T[-\xi\omega_n + j\omega_d]}$$

$$= e^{\left[\frac{-2\pi\xi}{\sqrt{1-\xi^2}}\frac{\omega_d}{\omega_s} + j2\pi\frac{\omega_d}{\omega_s}\right]}$$

$$|z| = e^{\left\{\frac{-2\pi\xi\,\omega_d}{(\sqrt{1-\xi^2})\omega_s}\right\}} \quad ; \quad \angle z = 2\pi\frac{\omega_d}{\omega_s}.$$

as $\omega_d \uparrow \Rightarrow |z| \downarrow$

$\omega_s \uparrow \Rightarrow |z| \uparrow$

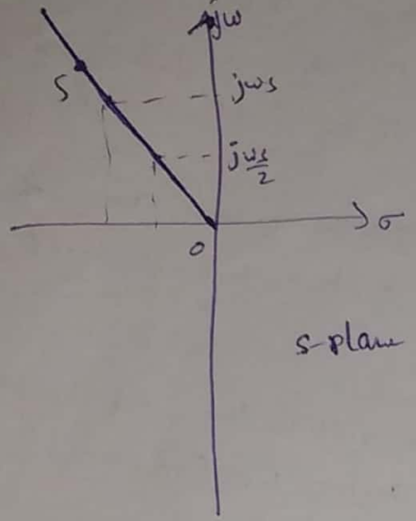
→ $\frac{\omega_d}{\omega_s}$ is called normalised frequency.

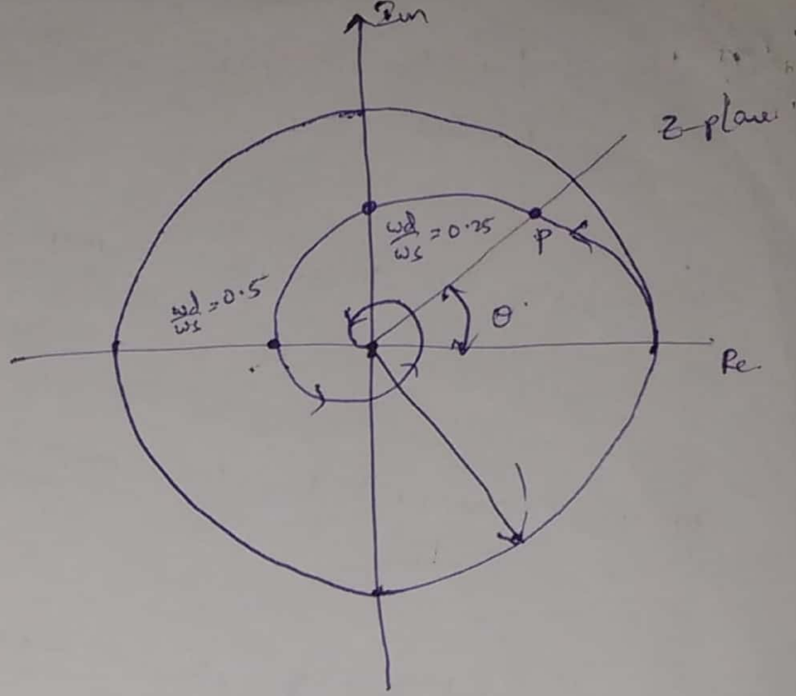
→ if ω_s is given ω_d can be find out. at any point on the path.

→ draw a line Joining origin and P.

→ measure the angle w.r.t +ve real axis

→ then obtain $2\pi\frac{\omega_d}{\omega_s} = \theta$.





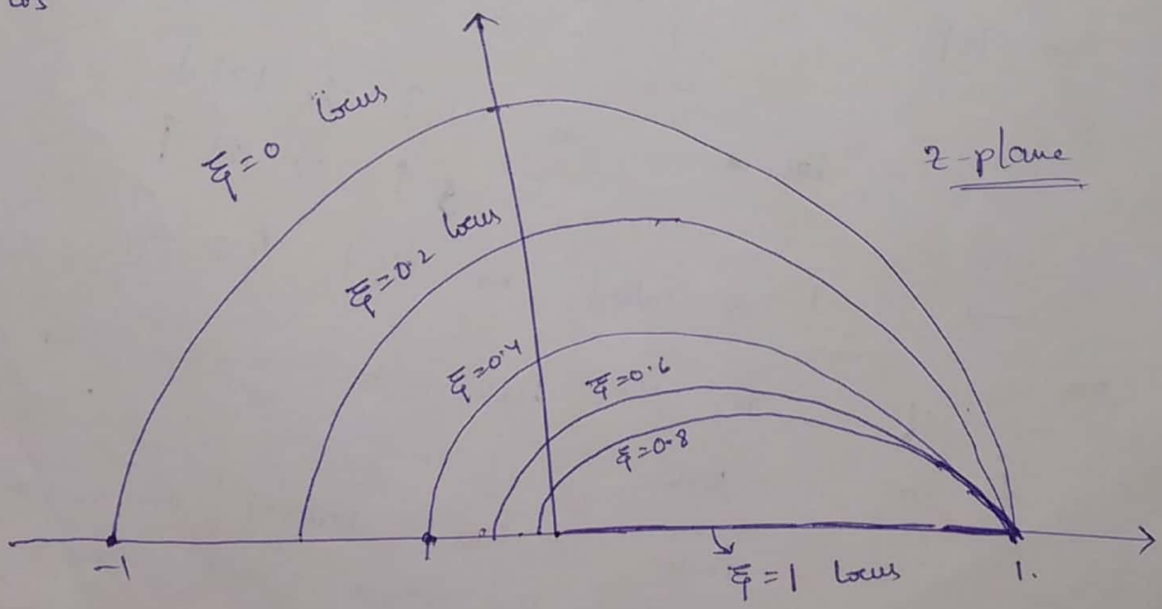
Spiral

~~If ωd is given~~

Constant damping ratio - line in s-plane	Spiral in z-plane
(i) 2nd (or) 3rd quadrant →	Spiral decays with in the unit circle.
(2) 1st ~~(or)~~ 4th quadrant →	Spiral grows outside the unit circle.

∴ $\frac{\omega_d}{\omega_s}$ is ~~positive~~.

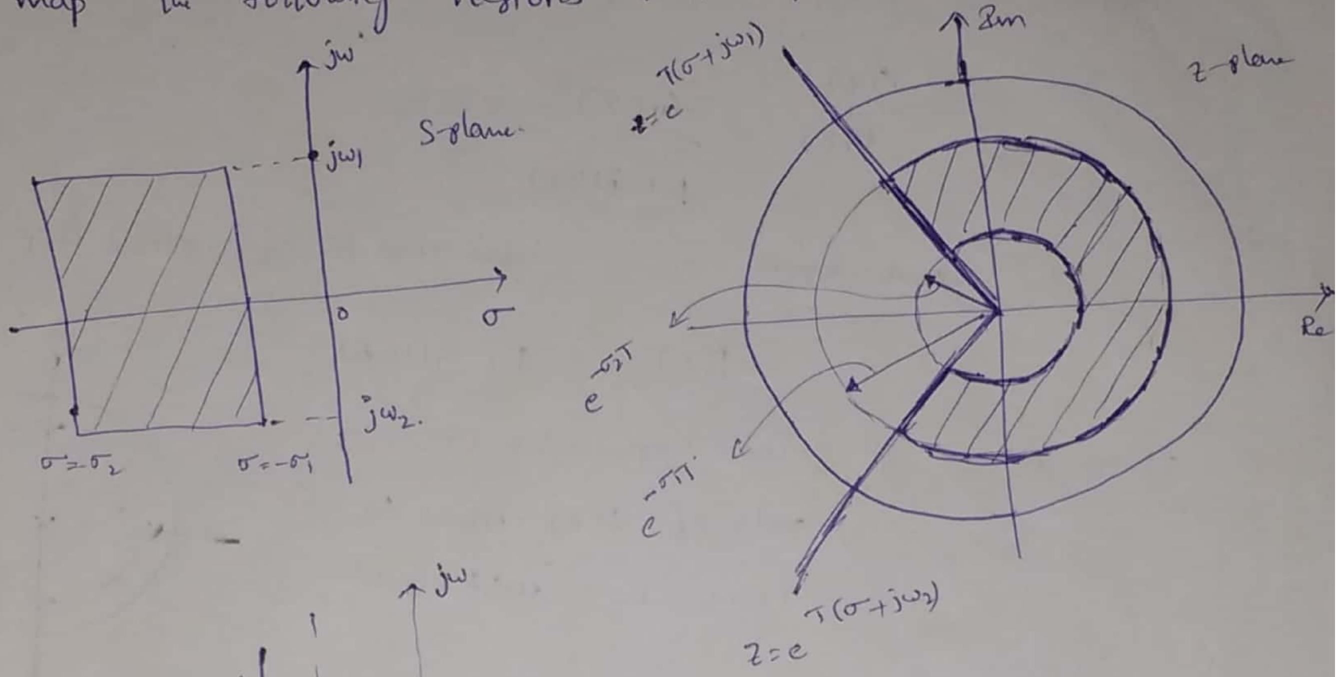
∴



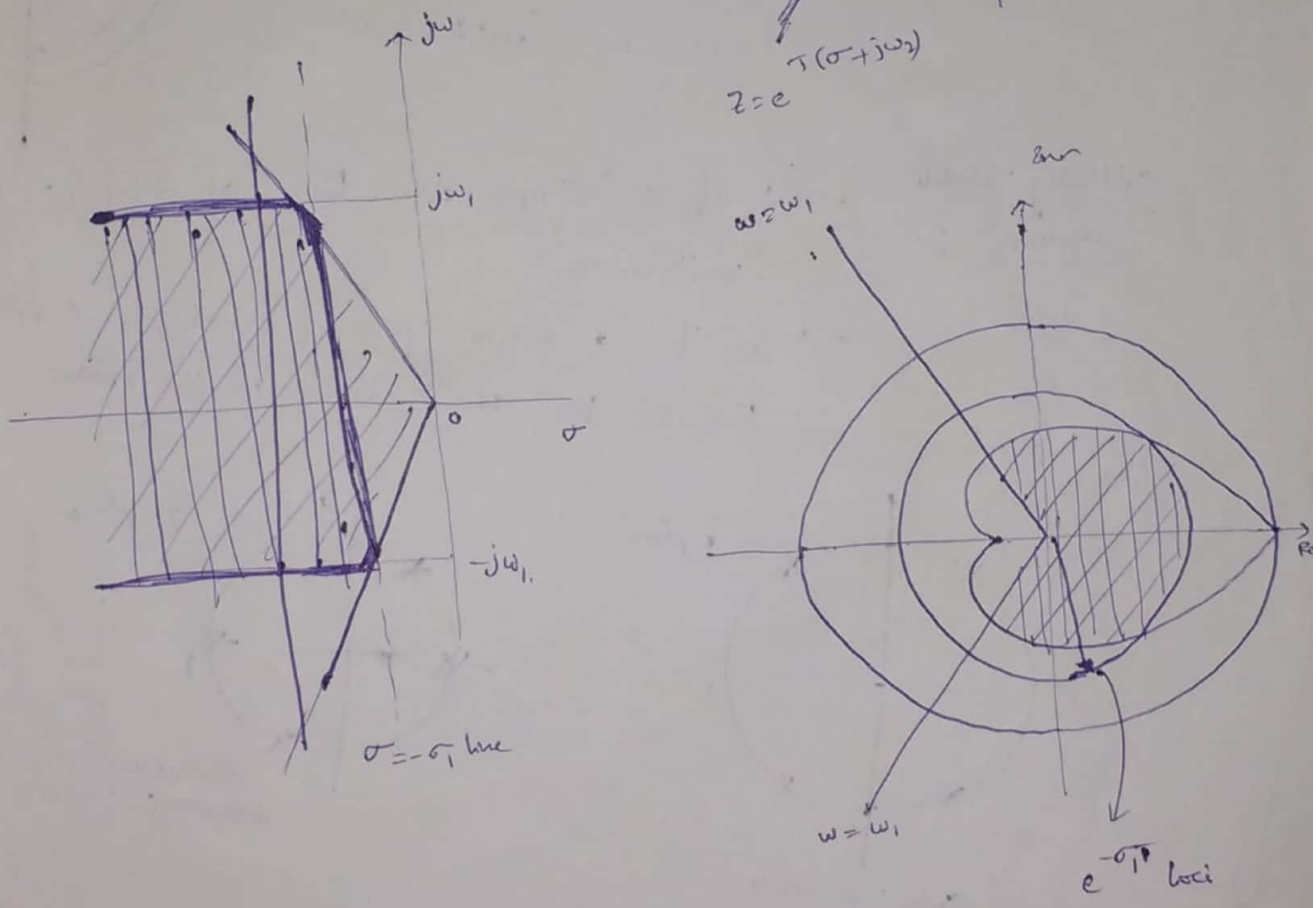
Problems

(a) map the following regions from s-plane to z-plane.

(1)



(2)



⇒ Stability Analysis (of L.T.I, SISO, Discrete-time Control Systems)

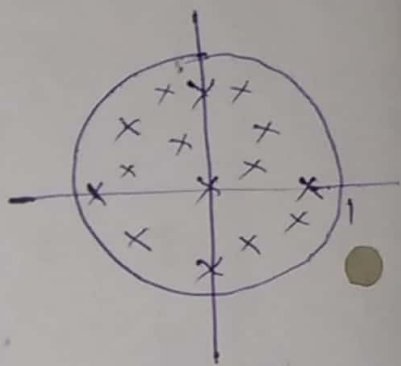
Consider a general closed loop P.T.F

$$\therefore \frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

the characteristic eqn = denominator of closed loop P.T.F

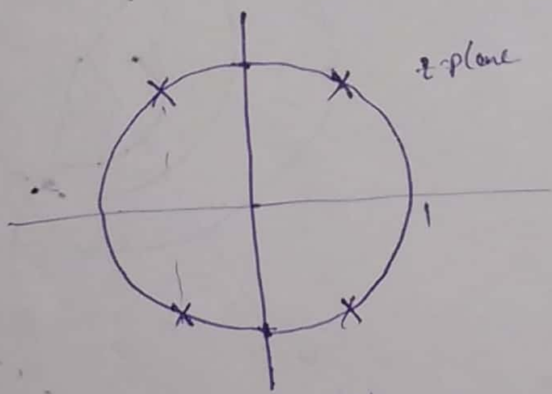
$$P(z) = 1 + GH(z)$$

Stable system → closed loop poles (or) roots of P(z) must lie inside the unit circle

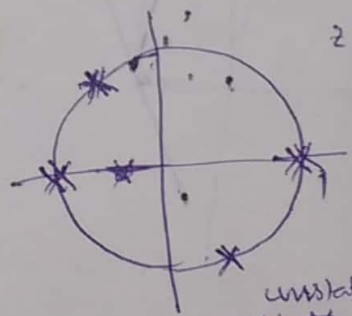


Critically stable system :- → ① if a simple pole lies at z=1 (i.e on the unit circle

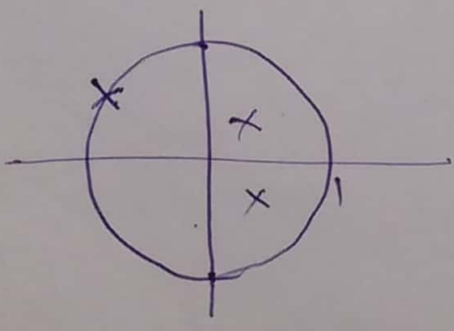
→ ② if a single pair of conjugate complex poles lies on the unit circle in the z plane.



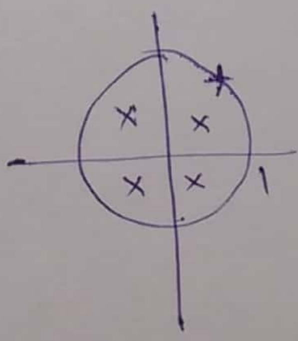
Critically stable



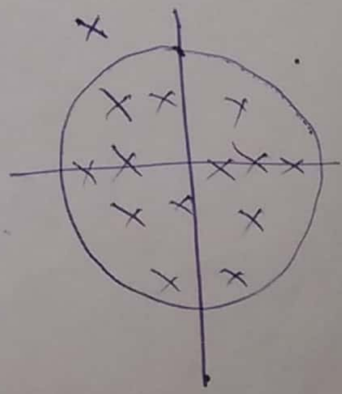
unstable
~~Critically~~ Stable



Critically Stable.



Critically Stable



Unstable

Unstable system :-

① if roots of $P(z)$ or closed loop poles lie outside the unit circle.

② if multiple roots (or multiple closed ~~pole~~ loop poles lie on the unit circle.

⇒ **Methods for Testing Absolute stability :-**

1. Jury's stability Test
2. Bilinear transformation coupled with Routh stability criterion

① Jury - stability Test

→ Gives the existence of any unstable roots (ie roots that lie outside the unit circle.)

→ The test Neither gives the location of unstable roots nor indicate the effect of parameter changes on the system stability.

→ let $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$; $a_0 > 0$

Jury table :-

Row	z^0	z^1	z^2	z^3	...	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	a_{n-3}	...	a_2	a_1	a_0
2.	a_0	a_1	a_2	a_3	...	a_{n-2}	a_{n-1}	a_n
3.	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}	...	b_1	b_0	
4.	b_0	b_1	b_2	b_3	...	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}	...	c_0		
6	c_0	c_1	c_2	c_3	...	c_{n-2}		
⋮	⋮							
2n-5	p_3	p_2	p_1	p_0				
2n-4	p_0	p_1	p_2	p_3				
2n-3	q_2	q_1	q_0					

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix} \quad k = 0, 1, 2, \ldots n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix} \quad k = 0, 1, 2 \ldots n-2$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix} \quad k = 0, 1, 2$$

Condition for stability :-

1. $|a_n| < a_0$;
2. $P(z)\big|_{z=1} > 0$
3. $P(z)\big|_{z=-1} \begin{cases} > 0 & \text{for } n \text{ even} \\ < 0 & \text{for } n \text{ odd.} \end{cases}$
4. $|b_{n-1}| > |b_0|$
   - $|c_{n-2}| > |c_0|$

     $\vdots$

     $|q_2| > q_0$

for a 4^{th} order polynomial :- $P(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$.

$$b_3 = \begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix} ; \quad b_2 = \begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix} ; \quad b_1 = \begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix} ; \quad b_0 = \begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix}$$

$$c_2 = \begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix} ; \quad c_1 = \begin{vmatrix} b_3 & b_1 \\ b_0 & b_2 \end{vmatrix} ; \quad c_0 = \begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix}.$$

array :-

	rows	z^0	z^1	z^2	z^3	z^4
	1	a_4	a_3	a_2	a_1	a_0
	2	a_0	a_1	a_2	a_3	a_4.
	3	b_3	b_2	b_1	b_0	
	4	b_0	b_1	b_2	b_3	
$(2n-3)$	5	c_2	c_1	c_0		

for a 3rd-order Polynomial.

$$P(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3.$$

	z^0	z^1	z^2	z^3
1	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3
$(2n-3) =$ 3	b_2	b_1	b_0	

$$b_2 = \begin{vmatrix} a_3 & a_0 \\ a_0 & a_3 \end{vmatrix}; \quad b_1 = \begin{vmatrix} a_3 & a_1 \\ a_0 & a_2 \end{vmatrix}; \quad b_0 = \begin{vmatrix} a_3 & a_2 \\ a_0 & a_1 \end{vmatrix}$$

Ex Consider the D.T unity-feedback control system (with $T=1$). whose open-loop P.T.F is

$$G(z) = \frac{k(0.3679z + 0.2642)}{(z - 0.3679)(z-1)}.$$

Determine the range of k for stability by use of Jury stability Test

Sol

$$C.L.\ P.T.F = \frac{G(z)}{1+G(z)} = \frac{k(0.3679z + 0.2642)}{z^2 + (0.3679k - 1.3679)z + 0.3679 + 0.2642k}$$

~~order of the~~ polynomial $P(z) = z^2 + (0.3679k - 1.3679)z + 0.3679 + 0.2642k.$

order = 2.

$\therefore$ No. of rows $= (2n-3) = \underline{1}$

$\therefore$ array

z^0	z^1	z^2
a_2	a_1	a_0

where

$a_0 = 1 \qquad a_2 = 0.3679 + 0.2642k$

For a stable system according to Jury stability,

① $|a_2| < a_0$

② $P(1) > 0$

③ $P(-1) > 0$ (∵ $n=2$ even number)

① ⇒ $|0.3679 + 0.2642k| < 1$ ⇒

⇒ $(0.3679 + 0.2642k) < 1$ ⇒ $k < 2.3925$; $-(0.3679 + 0.2642k) < 1$ ⇒ $k > -5.177$

② ⇒ $P(1) = \cancel{1} + (0.3679k - \cancel{1.3679}) + \cancel{0.3679} + 0.2642k > 0$

∴ $k > 0$.

③ ⇒ $P(-1) = 1 - (0.3679k - 1.3679) + 0.3679 + 0.2642k > 0$.

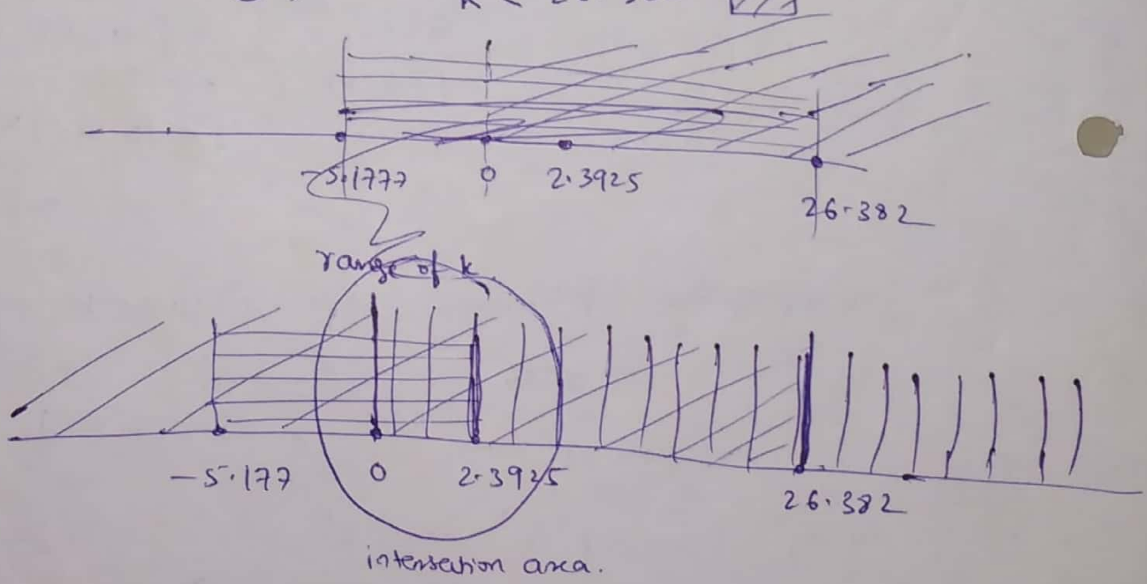
$= 26.382 > k$.

∴ k value must satisfy

① ⇒ $2.3925 > k > -5.1777$

② ⇒ $k > 0$

③ ⇒ $k < 26.382$



∴ k range is $0 < k < 2.3925$.

or $k \in (0, 2.392)$

Pb Using Jury stability Test obtain the stability of the following Discrete time systems. 6.7 (69)

(i) $z^3 + 3.3z^2 + 4z + 0.8 = 0$

H.W. (ii) $z^3 - 1.1z^2 - 0.1z + 0.2 = 0$ Jury

Sol (i)

$$P(z) = z^3 + 3.3z^2 + 4z + 0.8$$

Comparing to $P(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$

$a_3 = 0.8 \qquad a_2 = 4 \qquad a_1 = 3.3 \qquad a_0 = 1.$

Order of polynomial = $n = 3$.

$\therefore$ no. of rows = $2n - 3 = 3$.

	z^0	z^1	z^2	z^3
1	0.8	4	3.3	1
2	1	3.3	4	0.8
3.	-0.36	-0.1	-1.36.	

$b_2 = -0.36 \qquad b_1 = -0.1 \qquad b_0 = -1.36.$

Jury stability Test

① $|a_n| < a_0$ i.e. $a_3 < a_0$ satisfied.

② $P(1) > 0 \Rightarrow 1 + 3.3 + 4 + 0.8 > 0$ satisfied.

③ $P(-1) < 0 \Rightarrow -1 + 3.3 - 4 + 0.8 < 0$ satisfied.
($\because n = 3$ odd)

④ $|b_2| > |b_0|$. not satisfied.

$\therefore$ System is unstable.

Eg Examine the stability of the following characteristic equation.

$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08.$

$(z-0.8)(z+0.5)(z-0.5)(z-0.4) = 0$

Pr 2. Consider the following.

$$y(k) - 0.6y(k-1) - 0.81y(k-2) + 0.67y(k-3) - 0.12y(k-4) = x(k).$$

Determine the stability.

Sol. applying z-transform.

$$Y(z) - 0.6z^{-1}Y(z) - 0.81z^{-2}Y(z) + 0.67z^{-3}Y(z) - 0.12z^{-4}Y(z) = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 0.6z^{-1} - 0.81z^{-2} + 0.67z^{-3} - 0.12z^{-4}}$$

$$= \frac{z^4}{z^4 - 0.6z^3 - 0.81z^2 + 0.67z - 0.12}.$$

Comparing $a_0 = 1$ $a_1 = -0.6$ $a_2 = -0.81$ $a_3 = 0.67$ $a_4 = -0.12$.

Order of the system $= n = 4$.

$\therefore$ no. of rows $= 2n-3 = 5$.

	z^0	z^1	z^2	z^3	z^4
1	a_4 $\boxed{-0.12}$	a_3 $\boxed{0.67}$	a_2 $\boxed{-0.81}$	a_1 $\boxed{-0.6}$	a_0 $\boxed{1}$
2	a_0 $\boxed{1}$	a_1 $\boxed{-0.6}$	a_2 $\boxed{-0.81}$	a_3 $\boxed{0.67}$	a_4 $\boxed{-0.12}$
3	b_3 $\boxed{-0.9856}$	b_2 $\boxed{0.5196}$	b_1 $\boxed{0.9072}$	b_0 $\boxed{-0.5980}$	
4	b_0 $\boxed{-0.5980}$	b_1 $\boxed{0.9072}$	b_2 $\boxed{0.5196}$	b_3 $\boxed{-0.9856}$	
5	c_2 $\boxed{0.6138}$	c_1 $\boxed{0.0353}$	c_0 $\boxed{-0.5834}$		

Tests

1. $|a_4| < a_0$. Satisfied.
2. $P(1) \Rightarrow 1 - 0.6 - 0.81 + 0.67 - 0.12 = 0.14 > 0$ Satisfied.
3. $P(-1) \Rightarrow 1 + 0.6 - 0.81 - 0.67 - 0.12 = 0$

   i.e there is a root at $z = -1$.
4. $|b_3| > |b_0|$ Satisfied.
5. $|c_2| > |c_0|$ Satisfied

$\therefore$ critically stable.

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Stability analysis using Bilinear – transformation coupled with Routh's stability criterion

→ This method Requires transformation from z-plane to other Complex plane, ~~Called~~ say w-plane.

→ The bilinear – transformation is defined by

$$z = \frac{w+1}{w-1} \Rightarrow w = \frac{z+1}{z-1}.$$

mapping between z-plane & w-plane.

in z-plane		in w-plane.
Inside the unit circle	⟶	left half of w-plane.
Unit circle	⟶	Imaginary axis of w-plane
Outside the unit circle	⟶	Right half of w-plane

Steps

1. Substitute $\frac{w+1}{w-1}$ in place of z in the characteristic eqn.

then $P(z) \Rightarrow a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$ becomes

$$q(w) \Rightarrow a_0\left(\frac{w+1}{w-1}\right)^n + a_1\left(\frac{w+1}{w-1}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{w+1}{w-1}\right) + a_n = 0.$$

2. clear the fractions by multiplying the above q(w) both sides with $(w-1)^n$. then we obtain

$$Q(w) = b_0 w^n + b_1 w^{n-1} + \cdots + b_{n-1} w + b_n = 0.$$

3. Apply the Routh stability creterion in the Same manner as in Continuous – time systems.

⇒ Comparision between Jury stability and stability by Bilinear- transformation :-

① Computationally Jury stability test is simpler.

② the no. of poles inside the unit circle, on the unit Circle and outside the unit circle can't be found by Jury stability creterion where as it is possible by Bilinear - transformation.

Ex Given $P(z) \Rightarrow z^3 - 1.3z^2 - 0.08z + 0.24 = 0$. use Bilinear transformation for stability.

Sol
1. $\left(\frac{w+1}{w-1}\right)^3 - 1.3\left(\frac{w+1}{w-1}\right)^2 - 0.08\left(\frac{w+1}{w-1}\right) + 0.24 = 0.$

2. multiply with $(w-1)^3$.

$(w+1)^3 - (1.3)(w+1)^2(w-1) - 0.08(w+1)(w-1)^2 + 0.24(w-1)^3 = 0$

$w^3 + 1 + 3w^2 + 3w - 1.3\left[w^3 + w + 2w^2 - w^2 - 1 - 2w\right] - 0.08\left[w^3 + w - 2w^2 + w^2 + 1 - 2w\right]$

$+ 0.24\left[w^3 - 1 - 3w^2 + 3w\right] = 0$

$\therefore \quad Q(w) \Rightarrow -0.14w^3 + 1.06w^2 + 5.10w + 1.98 = 0$

$\Rightarrow w^3 - 7.571w^2 - 36.43w - 14.14 = 0$

Routh's Creterion :-

w^3	1	-36.43
w^2	-7.571	-14.14.
w^1	-38.3	0
w^0	-14.14	

sign change

one root → R.H of w-plane ⇒ outside the unit circle.

∴ unstable.

Ex 2 $4z^3 - 4z^2 - 7z - 2 = 0$