unit - 3

State space Representation

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Advantages

* Analysis of control system through root locus and frequency response methods are useful for SISO (single input single output) systems and LTI (Linear Time invariant) system.
* The P.T.F is only defind for linear system and fails for non-linear system.
* P.T.F is not applicable for optimal control system.
* The state space representation system is useful anytype of system (LTI, LTV, NLTV with MIMO system).
* The state space representation plays a vital roal in modern control engineering [optimal, adoptoable, robust control]
* In this representation the parameters of the plants & other elements are directly involved there will be a great possible, to see physical inside of the system.

State and state variables :-

* The state variables of a dynamic system are the variables making a smallest set of variables that determine the state of the dynamic system.
* If at least n variables $(x_1, x_2, x_3 \cdots x_n)$, complety describe the system, so that once the i/p is given for at a time $t \geq t_0$ and the knowledge of the initial state

at $t = t_0$ is specified future state of the system is completly determined then such "n" variables or a set of state variables.

* The no of state variables are unique but the set of state variables are not unique.

→ State space :-

* It is a "n" dimensional space whoose coordinate axis are the $x_1, x_2, x_3 ------ x_n$

→ State space representation :-

(a) L.T.I (Linear time invariant system) :-

$$\underset{n\times 1}{X(K+1)} = \underset{n\times n}{G}\,\underset{n\times 1}{X(K)} + \underset{n\times r}{H}\,\underset{r\times 1}{U(K)}$$

r - i/p state variables

m - o/p " "

$$\underset{m\times 1}{Y(K)} = \underset{m\times n}{C}\,\underset{n\times 1}{X(K)} + \underset{m\times r}{D}\,\underset{r\times 1}{U(K)}$$

(b) L.T.V (Linear time variant systems) :-

$$X(K+1) = G(K)\,X(K) + H(K)\,U(K)$$

$$Y(K) = C(K)\,X(K) + D(K)\,U(K)$$

where

X(K) - is state vector of dimensions of $n\times 1$ matrix

U(K) - input vector (r no. of input) $r\times 1$

Y(K) - output vector (m no. of output) $m\times 1$

G - state matrix / system matrix

H - input matrix.

C - output matrix

D - Direct transmission matrix

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(c) NLTI

$$x(k+1) = f(x(k), u(k))$$

$$y(k) = g(x(k), u(k))$$

* LTI can be modelled by the following four types :-

→ Controllable canonical form (Direct)

→ Observed canonical form (nested)

→ Diagonal canonical form

→ Jordan canonical form

$\left.\begin{matrix} \text{Diagonal canonical form} \\ \text{Jordan canonical form} \end{matrix}\right\}$ using partial fraction.

Consider the system equation

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \dots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \dots + b_n u(k-n)$$

a → Controllable canonical form :-

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n}$$

system representation

$$x(k+1) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}_{n \times n}}_{G} x(k) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times n}}_{H} u(k)$$

$$y(k) = \begin{bmatrix} b_n - b_0 a_n & b_{n-1} - b_0 a_{n-1} & \cdots & b_1 - b_0 a_1 \end{bmatrix} x(k) + b_0 u(k)$$

SISO system

→ Observable canonical form [nested programming] :-

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots\cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots\cdots + a_n z^{-n}}$$

$$\Rightarrow Y(z)\left[1 + a_1 z^{-1} + \cdots\cdots + a_n z^{-n}\right] = U(z)\left[b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}\right]$$

$$\Rightarrow Y(z) - b_0 U(z) + z^{-1}\left[a_1 Y(z) - b_1 U(z)\right] + z^{-2}\left[a_2 Y(z) - b_2 U(z)\right]$$

$$+ \cdots\cdots + z^{-n}\left[a_n Y(z) - b_n U(z)\right] = 0$$

$$\Rightarrow Y(z) - b_0 U(z) + z^{-1}\Big\{(a_1 Y(z) - b_1 U(z)) + z^{-1}\left[a_2 Y(z) - b_2 U(z)\right]$$

$$+ z^{-2}(a_3 Y(z) - b_3 U(z))\Big\} + \cdots\cdots + z^{-n+1}(a_n Y(z) - b_n U(z))$$

$$= 0$$

$$\Rightarrow Y(z) = b_0 U(z) + z^{-1}\Big\{\left[b_1 U(z) - a_1 Y(z)\right] + z^{-1}\Big\{(b_2 U(z) - a_2 Y(z))$$

$$+ \cdots\cdots + z^{-1}\left[b_n U(z) - a_n Y(z)\right]\Big\}\Big\}$$

define state variables

$$x_1(z) = z^{-1}\left[b_n U(z) - a_n Y(z)\right]$$

$$x_2(z) = z^{-1}\left[b_{n-1} U(z) - a_{n-1} Y(z)\right] + z^{-1}\left[b_n U(z) - a_n Y(z)\right]$$

$$x_2(z) = z^{-1}\Big\{\left[b_{n-1} U(z) - a_{n-1} Y(z)\right] + x_1(z)\Big\}$$

$$x_n(z) = z^{-1}\left[b_1 U(z) - a_1 Y(z) + X_{n-1}(z)\right]$$

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$$\boxed{\begin{aligned} Y(z) &= b_0 U(z) + X_n(z) \\ x_n(z) &= z^{-1}\left[b_1 U(z) - a_1 Y(z) + X_{n-1}(z)\right] \end{aligned}}$$

$$\Rightarrow z X_1(z) = b_n U(z) - a_n\left[b_0 U(z) + x_n(z)\right]$$

$$= U(z)\left[b_n - a_n b_0\right] - a_n x_n(z)$$

$$\Rightarrow z x_2(z) = b_{n-1} U(z) - a_{n-1} b_0 U(z) - a_{n-1} x_{n-1}(z)$$

$$\Rightarrow b_{n-1} U(z) - a_{n-1} b_0 U(z) - a_{n-1} x_n(z) + \left[b_n U(z) - a_n\left[b_0 U(z) + x_n(z)\right]\right]$$

$$\Rightarrow U(z)\left[b_{n-1} - a_{n-1} b_0 + b_n - a_n b_0\right] - X_n(z)\left[a_{n-1} + a_n\right]$$

$$\Rightarrow z X_3(z) = U(z)\left[b_{n-2} - a_{n-2} b_0 + b_{n-1} - a_{n-1} b_0 + b_n - a_n b_0\right] - X_n(z)\left[a_{n-2} + a_{n-1} + a_n\right]$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 0 & 1 & & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & & 1 & 0 & -a_2 \\ 0 & 0 & & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \\ \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} b_n - b_0 a_n \\ b_{n-1} - b_0 a_{n-1} \\ \\ \\ \\ b_1 - b_0 a_1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

Let G, H, C, D are the state space matrices in Controllable canonical form.

G', H', C', D' are the state space matrices in Observed canonical form.

$$G' = G^T \;;\; H' = C^T \;;\; C' = H^T \;;\; D' = D$$

If we know one form (controllable canonical form), we can write Observed canonical form.

→ Diagonal canonical form :-

Let the closed loop transfer function

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

* Factorize the denominator to find the poles.
* If the poles are distinct, we can write in diagonal canonical form.
* Perform partial fractions.

We get

$$\frac{Y(z)}{U(z)} = b_0 + \frac{C_1}{z - P_1} + \frac{C_2}{z - P_2} + \cdots + \frac{C_n}{z - P_n}$$

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$$C_1 = \lim_{Z \to P_1} (Z - P_1) \cdot \frac{Y(z)}{U(z)}$$

$$C_i = \lim_{Z \to P_i} (Z - P_i) \frac{Y(z)}{U(z)}$$

$i = 1, 2, 3 \dots n$

$$X(k+1) = \begin{bmatrix} P_1 & 0 & 0 & \dots & 0 \\ 0 & P_2 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & & P_n \end{bmatrix}_{n\times n} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}_{n\times 1} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1} u(k)$$

$$Y(k) = \begin{bmatrix} C_1 & C_2 & \dots & C_n \end{bmatrix}_{1\times n} \begin{bmatrix} x_1(k) \\ x_n(k) \end{bmatrix}_{n\times 1} + b_0 u(k)$$

→ Jordan canonical form :-

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_n z^{-n}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}}$$

If the system has the multiple poles of the order "q" we cann't write in diagonal canonical form but we can represent the system in jordan canonical form.

Let

$Z = P_1$ is a pole of order q then the partial fractions will results in

$$\frac{Y(z)}{U(z)} = b_0 + \frac{C_1}{(Z-P_1)^q} + \frac{C_2}{(Z-P_2)^{q-1}} + \dots + \frac{C_q}{Z-P_1} + \frac{C_{q+1}}{} + \dots + \frac{C_n}{Z-P_n}$$

$$x(k+1) = \left[\begin{array}{cccc|ccc} p_1 & 1 & 0 \cdots & 0 & & & \\ 0 & p_1 & 1 \cdots & 0 & & 0 & \\ \vdots & & & & & & \\ 0 & 0 & 0 \cdots & p_1 & & & \\ \hline & & & & p_{q+1} & 0 \cdots & 0 \\ & 0 & & & & & \\ & & & & 0 & 0 \cdots & p_n \end{array}\right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_q(k) \\ x_{q+1}(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

For a given transfer function write CCF, OCF, DCF.

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2+1.3z+0.4}$$

sol :-

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2+1.3z+0.4} \times \frac{z^{-2}}{z^{-2}}$$

controllable canonical form :

$$\frac{Y(z)}{z^{-1}+z^{-2}} = \frac{U(z)}{1+1.3z^{-1}+0.4z^{-2}} = Q(z)$$

$$\Rightarrow U(z) = Q(z) + 1.3z^{-1}Q(z) + 0.4z^{-2}Q(z) \quad \text{--- (a)}$$

$$X_1(z) = z^{-1}Q(z) \Rightarrow zX_1(z) = Q(z) \quad \text{--- (1)}$$

$$X_2(z) = z^{-2}Q(z) \Rightarrow zX_2(z) = X_1(z) \quad \text{--- (2)}$$

apply I.Z.T to equation

$$x_2(k+1) = x_1(k) \quad \text{--- (3)}$$

From 1 & 2 & ⓐ

us

$$zX_1(z) = U(z) - 1.3z^{-1}Q(z) - 0.4z^{-2}Q(z)$$

$$= U(z) - 1.3X_1(z) - 0.4X_2(z)$$

$$x_1(k+1) = u(k) - 1.3x_1(k) - 0.4x_2(k) \quad \text{—— (4)}$$

From 3 & 4

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1.3 & -0.4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$Y(z) = z^{-1}Q(z) + z^{-2}Q(z)$$

$$= X_1(z) + X_2(z)$$

Inverse z. Transform

$$y(k) = x_1(k) + x_2(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + 0$$

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if $X_1(z) = z^{-2}Q(z)$

$X_2(z) = z^{-1}Q(z) \Rightarrow zX_2(z) = Q(z)$. then

$\therefore$ $z\cancel{X(z)} =$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

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O.C.F.:- $\frac{Y(z)}{U(z)} = \frac{z^{-1}+z^{-2}}{1+1.3z^{-1}+0.4z^{-2}}$

$Y(z) + 1.3z^{-1}Y(z) + 0.4z^{-2}Y(z) = z^{-1}U(z) + z^{-2}U(z)$

$Y(z) = z^{-1}[U(z) - 1.3Y(z)] + z^{-2}[U(z) - 0.4Y(z)]$.

$Y(z) = z^{-1}[U(z) - 1.3Y(z) + z^{-1}[U(z) - 0.4Y(z)]]$

let $z^{-1}[U(z) - 1.3Y(z) + z^{-1}(U(z) - 0.4Y(z))] = X_1(z)$ — (1)

$\therefore Y(z) = X_1(z)$ — (2)

and. $z^{-1}[U(z) - 0.4Y(z)] = X_2(z)$ — (3)

$\therefore$ (1), (3) $\Rightarrow$ $z^{-1}[U(z) - 1.3Y(z) + X_2(z)] = X_1(z)$ — (4)

(2), (4) $\Rightarrow$ $zX_1(z) = U(z) - 1.3X_1(z) + X_2(z)$

$x_1(k+1) = -1.3x_1(k) + x_2(k) + u(k)$ — (A)

(2), (3) $\Rightarrow$ $zX_2(z) = U(z) - 0.4X_1(z)$

$x_2(k+1) = -0.4x_1(k) + u(k)$ — (B)

(2) $\Rightarrow$ $y(k) = x_1(k)$ — (C)

(A), (B) $\Rightarrow$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1.3 & 1 \\ -0.4 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

2) $$\frac{Y(z)}{U(z)} = \frac{z^{-1}(1+z^{-1})}{(1+0.5z^{-1})(1-0.5z^{-1})}$$ 46

$$= \frac{z^{-1}+z^{-2}}{1-0.25\,z^{-2}}$$

(a) controllable canonical form

$$\Rightarrow \frac{Y(z)}{z^{-1}+z^{-2}} = \frac{U(z)}{1-0.25z^{-2}} = Q(z)$$

$$\Rightarrow U(z) = Q(z) - 0.25\,z^{-2}\,Q(z)$$

$$x_1(z) = z^{-1}Q(z)$$

$$x_2(z) = z^{-2}Q(z) \Rightarrow z^{2}x_2(z) = Q(z)$$

$$x_2(K+1) = x_1(K)$$

$$z x_1(z) = U(z) - 0.25\,x_2(z)$$

$$U(z) = Q(z) - 0.25\,z^{-2}Q(z)$$ assume z^{-1}

$$\Rightarrow Q(z) = U(z) + 0.25\,z^{+2}x(z)$$

$$x_1(z) = z^{-2}Q(z) \Rightarrow z^{2}x_1(z) = Q(z)$$

Inverse z-Transform

$$x_2(K+1) = x_1(K) \quad \text{——— (1)}$$

$$x_1(z) = z^{-1}Q(z)$$

$$z x_1(z) = U(z) - 0.25\,x_2(z)$$

$$Q(z) = U(z) + 0.25\,z^{2}\,x_2(z)$$

$$x_1(k+1) = u(k) + 0.25\, x_2(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$Y(z) = z^{-1}Q(z) + z^{-2}Q(z) = X_1(z) + X_2(z)$$

Inverse z. Transform.

$$y(k) = x_1(k) + x_2(k)$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + 0$$

Observable cananical form :–

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} X(k)$$

(c) Diagonal form $\dfrac{Y(z)}{U(z)} = \dfrac{z^{-1}(1+z^{-1})}{(1+0.5z^{-1})(1-0.5z^{-1})} = \dfrac{(1/z)(1+1/z)}{\left(1+\frac{0.5}{z}\right)\left(1-\frac{0.5}{z}\right)} = \dfrac{\dfrac{(z+1)}{\cancel{z^2}}}{\dfrac{(z+0.5)(z-0.5)}{\cancel{z^2}}}$

$$\therefore \frac{Y(z)}{U(z)} = \frac{A}{z+0.5} + \frac{B}{z-0.5}$$

$$\frac{Y(z)}{U(z)} = \frac{1.5}{z-0.5} - \frac{0.5}{z+0.5}$$

$$A = \left.\frac{z+1}{z-0.5}\right|_{z=-0.5} = \frac{-0.5+1}{-0.5-0.5} = \frac{0.5}{-1.0}$$

$$A = -0.5$$

$$B = \left.\frac{z+1}{z+0.5}\right|_{z=0.5} = \frac{1.5}{1}$$

$$\therefore \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

→ Solution of linear time invariant discrete time system :-

Consider

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$$\left.\begin{aligned} X(k+1) &= G X(k) + H U(k) \\ Y(k) &= C X(k) + D U(k) \end{aligned}\right\} \text{--- a}$$

sol :-

$$X(1) = G X(0) + H U(0)$$

$$X(2) = G X(1) + H U(1)$$

$$= G\left[G X(0) + H U(0)\right] + H U(1)$$

$$X(2) = G^2 X(0) + G H U(0) + H U(1)$$

$$\vdots$$

$$X(k) = G^k X(0) + \sum_{j=0}^{k-1} G^{k-j-1} H U(j) \quad \text{--- (1)}$$

$$Y(k) = C\left[G^k X(0) + \sum_{j=0}^{k-1} G^{k-j-1} H U(j)\right] + D U(k)$$

$$= C G^k X(0) + C \sum_{j=0}^{k-1} G^{k-j-1} H U(j) + D U(k) \quad \text{--- 2}$$

Here in equation - 1.

$\phi(k) = G^k$ is called state transition matrix. or fundamental matrix

$$\therefore \boxed{X(k) = \phi(k) X(0) + \sum_{j=0}^{k-1} \phi(k-j-1) H U(j)}$$

⇒ Z - Transform approach to find state transition methods :-

$$X(k+1) = G X(k) + H U(k)$$

apply Z - Transform

$$zX(z) - zX(0) = G X(z) + H U(z)$$

$$zX(z) - G X(z) = zX(0) + H U(z)$$

$$X(z)\left[zI - G\right] = zx(0) + HU(z)$$

$$X(z) = \left[zI - G\right]^{-1}\left[zx(0) + HU(z)\right]$$

apply inverse z-Transform.

$$x(k) = I.Z.T\left\{\left[\left[zI - G\right]^{-1} z\right] x(0) + (zI-G)^{-1} H \cdot U(z)\right\}$$

compare above equation with eq ①

then

$$\phi(k) = g(k) = I.Z.T\left[(zI - G)^{-1} z\right]$$

↓

state transition matrix.

⇒ Method to find $[zI - G]^{-1}$:=

If the order of the matrix is ≥ 3 then $[zI - G]^{-1}$ ~~inverse~~ of any matrix is adjoint of that matrix by determinant of that matrix.

$$[zI - G]^{-1} = \frac{adj\,[zI - G]}{|zI - G|}$$

let

$$|zI - G| = z^{n} + a_1 z^{n-1} + a_2 z^{n-2} + \cdots\cdots + a_n \quad \text{— ①}$$

$$adj\,[zI - G] = I z^{n-1} + H_1 z^{n-2} + H_2 z^{n-3} + \cdots\cdots + H_{n-1}$$

$$H_1 = G + a_1 I$$

$$H_2 = GH_1 + a_2 I$$

$$H_3 = GH_2 + a_3 I$$

$$\vdots$$

$$H_{n-1} = GH_{n-2} + a_{n-1} I$$

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If $z = 3\times3$ matrix

$$|zI-G| = z^3+a_1z^2+a_2z+a_3$$

$$\text{adj}\,[zI-G] = Iz^2+H_1z+H_2$$

$$H_1 = G+a_1I \; ; \quad H_2 = GH_1+a_2I$$

* Calculation of P.T.F from state space form :-

consider

$$X(K+1) = GX(K)+HU(K) \quad \text{——} ①$$

$$Y(K) = CX(K)+DU(K) \quad \text{——} ②$$

apply Z-Transform to equation 1,2

$$\Rightarrow \quad zX(z) - \cancel{zX(0)} = GX(z)+HU(z)$$

$$Y(z) = CX(z)+DU(z)$$

$$\Rightarrow \quad X(z)\left[zI-G\right] = HU(z).$$

$$X(z) = \left[zI-G\right]^{-1}HU(z) \quad \text{——} ③$$

$$Y(z) = CX(z)+DU(z) \quad \text{——} ④$$

from 3, 4

$$Y(z) = C\left[zI-G\right]^{-1}HU(z)+DU(z)$$

$$= \left[C(zI-G)^{-1}H+D\right]U(z)$$

$$\boxed{P.T.F = \frac{Y(z)}{U(z)} = C(zI-G)^{-1}H+D}$$

Controllability and Observability :-

State Controllability :-

A control system is controllable or completly state controllable. If every state variable can be controlled in a finite time by some ~~un~~constrained or unbalanced control signal.

Conditions for state controllability :-

$$X(K+1) = GX(K) + HU(K)$$

where G is a nxn sized matrix and H is a nxr matrix.

Define controllability matrix as a

$$M_c = \left[\underset{n\times r}{H} \;\vdots\; \underset{n\times r}{GH} \;\vdots\; G^2H \;\vdots\; - - - - \; G^{n-1}H \right]_{n\times nr}$$

The condition for state controllability is the rank of Mc

Mc is must be n $\longrightarrow$ order of the matrix.

State Observability :-

The system is completly observable if given the o/p Y(K) over the finite no. of sampling periods, it is possible to determine the initial state vector, X(0).

Define the observability matrix as

$$M_{ob} = \left[c^* \;\vdots\; G^* c^* \;\vdots\; G^{*2} c^* \;\vdots\; \ldots\ldots \; G^{*n-1} c^* \right] \text{(or)}.$$

$$M_{ob} = \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix}$$

Where * => conjugate transpose.

The condition for observability is the rank of Mob must be equal to n.

Output controllability :-

Complete state controllability is neither necessary nor sufficient for controlling of the system.

→ The system defind by

$$X(K+1) = G\,X(K) + H\,U(K)$$

$$Y(K) = C\,X(K) + D\,U(K)$$

is completly state controllability implies the complete output controllability if and only if the rank of

$$\text{rank}\begin{bmatrix} D \,\vdots\, CH \,\vdots\, CGH \,\vdots\, CG^2H \,\vdots\, \ldots\ldots\; CG^{n-1}H \end{bmatrix} = M$$

M – no. of outp

Problem:-

$$X(K+1) = G\,X(K) + H\,U(K)$$

$$y(K) = C\,U(K)$$

where

$$G = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 2 \end{bmatrix} ; \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

i) controllable   ii) Observalle.

Sol :-

$$M_c = \left[H \,;\, GH \,;\, G^2H \;\ldots\ldots\; G^{n-1} \right]$$

$n = 2$

$$M_c = \left[H \,;\, GH \right] \qquad GH = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & -7 \end{bmatrix} \qquad = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

$$|M_c| = -7 - 4 \neq 0$$

ii) Observability :-

$$M_{ob} = \left[c^* \mid G^* c^* \,;\, G^{*2} c^* \;\ldots\ldots\; G^{*n-1} c^* \right]$$

c^* – conjugate transpose of c

$$= \begin{bmatrix} 1 & 1 \end{bmatrix}^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$G^* = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 \\ 1 & -3 \end{bmatrix} \;; \quad G^* c^* = \begin{bmatrix} 0 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$M_{ob} = \left[c^* \,;\, G^* c^* \right] = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

$$|M_{ob}| = -2 + 1 = -1 \neq 0$$

rank = 2 ; no. of state variables = 2 the system is observable.

→ Obtain the P.T.F from a given system

Ex.

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$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 5 \\ -1 & 2 & 4 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} ; \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} ; \quad D = 0$$

Sol :- $P.T.F = C\left[zI - G\right]^{-1} H + D$

$$\left[zI - G\right]^{-1} = \frac{\text{adj}\left[zI - G\right]}{|zI - G|}$$

$$zI - G = \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 5 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} z-1 & 0 & 0 \\ 0 & z-3 & -5 \\ 1 & -2 & z-4 \end{bmatrix}$$

$$|zI - G| = (z-1)\left[(z-3)(z-4) - 10\right]$$

$$= (z-1)\left[z^2 - 7z + 2\right]$$

$$= z^3 - 7z^2 + 2z - z^2 + 7z - 2$$

$$= z^3 - 8z^2 + 9z - 2.$$

$$a_1 = -8 ; \quad a_2 = 9 ; \quad a_3 = -2$$

$$\text{adj}\left[zI - G\right] = Iz^{n-1} + H_1 z^{n-2} + H_2$$

$$= Iz^2 + H_1 z + H_2$$

$$H_1 = G + a_1 I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 5 \\ -1 & 2 & 4 \end{bmatrix} + \begin{bmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 5 \\ -1 & 2 & -4 \end{bmatrix}$$

$$H_2 = GH_1 + a_2 I$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 5 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 5 \\ -1 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 & 0 \\ -5 & -5 & -5 \\ 3 & -2 & -6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -5 & 4 & -5 \\ 3 & -2 & 3 \end{bmatrix}$$

$$\operatorname{adj}[zI - G] = \begin{bmatrix} z^2-7z+2 & 0 & 0 \\ -5 & z^2-5z+4 & 5z-5 \\ -z+3 & 2z-2 & z^2-4z+3 \end{bmatrix}$$

$$[zI - G]^{-1} = \frac{1}{z^3 - 8z^2 + 9z - 2} \begin{bmatrix} z^2-7z+2 & 0 & 0 \\ -5 & z^2-5z+4 & 5z-5 \\ -z+3 & 2z-2 & z^2-4z+3 \end{bmatrix}$$

$$P.T.F = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \frac{1}{z^3-8z^2+9z-2} \begin{bmatrix} z^2-7z+2 & 0 & 0 \\ -5 & z^2-5z+4 & 5z-5 \\ -z+3 & 2z-2 & z^2-4z+3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Pro Obtain the state transition matrix of the following

$$x(k+1) = Gx(k) + Hu(k) \quad (51)$$

$$y(k) = c\,x(k) + 0 ; \quad \text{where}$$

$$G = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 1 \end{bmatrix} ; \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

find the P.T.F

sol :-

$$[zI - G] = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} z & -1 \\ 2 & z+2 \end{bmatrix}$$

$$[zI - G]^{-1} = \frac{1}{z^2+2z+2} \begin{bmatrix} z+2 & 1 \\ -2 & z \end{bmatrix}$$

state transition matrix $\varphi(k)$:-

$$\text{I. Z.T of } \left[(zI - G)^{-1} z \right] :$$

$$\Rightarrow \text{I.Z.T} \begin{bmatrix} \dfrac{z(z+2)}{z^2+2z+2} & \dfrac{z}{z^2+2z+2} \\[2ex] \dfrac{-2z}{z^2+2z+2} & \dfrac{z^2}{z^2+2z+2} \end{bmatrix}$$

Properties of state transition matrix

$\phi(k) = G^k$ = state transition matrix = I.Z.T $[z(zI-A)^{-1}]$

① $\phi(0) = I$

② $\phi(k_1+k_2) = G^{k_1+k_2} = G^{k_1} \cdot G^{k_2} = \phi(k_1) \cdot \phi(k_2)$

③ $\phi(-k) = G^{-k} = [G^k]^{-1} = \phi^{-1}(k)$

$\therefore \quad \phi^{-1}(-k) = \phi(k)$.

Problems: Consider the D.C.S state eqn, $x(k+1) = Gx(k) + Hu(k)$

$$G = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Determine Controllability & ~~observability~~

② Controllability & observability

a $G = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. $H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = [1 \quad 1]$

b $G = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}$ $H = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $C = [1 \quad 1]$

c $G = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$ $H = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $C = [1 \quad 0]$

d $G = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

e $G = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$ $H =$.

f. $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ $[1 \quad 5]$ observability

Pro $G(z) = \frac{Y(z)}{U(z)} = \frac{z^2+2z+1}{z^3+2z^2+z+0.5}$ obtain ① C.C.F ② O.C.F

3.22

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C.C.F

Sol $\frac{Y(z)}{U(z)} = \frac{z^{-1}+2z^{-2}+z^{-3}}{1+2z^{-1}+z^{-2}+0.5z^{-3}}$

$\frac{Y(z)}{z^{-1}+2z^{-2}+z^{-3}} = \frac{U(z)}{1+2z^{-1}+z^{-2}+0.5z^{-3}} = Q(z)$

$\therefore \frac{Y(z)}{z^{-1}+2z^{-2}+z^{-3}} = Q(z)$ — ①

$\frac{U(z)}{1+2z^{-1}+z^{-2}+0.5z^{-3}} = Q(z)$ — ②

$\Rightarrow$ ② $\Rightarrow$ $U(z) = Q(z) + 2z^{-1}Q(z) + z^{-2}Q(z) + 0.5z^{-3}Q(z)$ — ③

Let
$X_1(z) = z^{-3}Q(z)$
$X_2(z) = z^{-2}Q(z)$
$X_3(z) = z^{-1}Q(z)$

$\Rightarrow$ $\boxed{zX_3(z) = Q(z)}$ — ④

$zX_2(z) = z^{-1}Q(z) = X_3(z)$.

$\therefore \boxed{zX_2(z) = X_3(z)} \Rightarrow x_2(k+1) = x_3(k)$ — (A)

$zX_1(z) = z^{-2}Q(z) = X_2(z)$

$\boxed{zX_1(z) = X_2(z)} \Rightarrow x_1(k+1) = x_2(k)$. — (B)

from ③ $Q(z) = U(z) - 2z^{-1}Q(z) - z^{-2}Q(z) - 0.5z^{-3}Q(z)$ — ⑤

from ④, ⑤ $zX_3(z) = U(z) - 2X_3(z) - X_2(z) - 0.5X_1(z)$

$\therefore x_3(k+1) = -0.5x_1(k) - x_2(k) - 2x_3(k) + u(k)$ — (C)

from (A), (B), (C)
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

from ① $Y(z) = z^{-1}Q(z) + 2z^{-2}Q(z) + z^{-3}Q(z) \Rightarrow Y(z) = X_3(z) + 2X_2(z) + X_1(z)$

$y(k) = x_3(k) + 2x_2(k) + x_1(k) \Rightarrow y(k) = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$.

Observable canonical form (O.C.F)

$$\frac{Y(z)}{U(z)} = \frac{z^{-1}+2z^{-2}+z^{-3}}{1+2z^{-1}+z^{-2}+0.5z^{-3}}$$

$$Y(z) + 2z^{-1}Y(z) + z^{-2}Y(z) + 0.5z^{-3}Y(z) = U(z)z^{-1} + 2z^{-2}U(z) + z^{-3}U(z)$$

$$Y(z) = z^{-1}(-2Y(z)+U(z)) + z^{-2}[-Y(z)+2U(z)] + z^{-3}[U(z)-0.5Y(z)].$$

$$= z^{-1}\left[(U(z)-2Y(z)) + z^{-1}(2U(z)-Y(z)) + z^{-2}[U(z)-0.5Y(z)]\right]$$

$$Y(z) = z^{-1}\left[(U(z)-2Y(z)) + z^{-1}\{(2U(z)-Y(z)) + z^{-1}(U(z)-0.5Y(z))\}\right] \quad \text{— (1)}$$

Lets define $z^{-1}[U(z)-0.5Y(z)] = X_3(z)$ & ~~...~~ — (2)

$$\Rightarrow \cancel{zX_3(z)} = \cancel{U(z)-0.5Y(z)}.$$

$$z^{-1}\left[2U(z)-Y(z) + z^{-1}(U(z)-0.5Y(z))\right] = X_2(z).$$

$$z^{-1}\left[2U(z)-Y(z) + X_3(z)\right] = X_2(z). \quad \text{— (3)}$$

$$\& \; z^{-1}\left[U(z)-2Y(z) + z^{-1}\left[2U(z)-Y(z) + z^{-1}\{U(z)-0.5Y(z)\}\right]\right] = X_1(z) \quad \text{— (4)}$$

$$\cancel{zX_1(z)} = \cancel{U(z)-2Y(z)} \quad \therefore Y(z) = X_1(z). \quad \text{— (5)}$$

(4), (5) $\Rightarrow$
$$zX_1(z) = U(z) - 2X_1(z) + X_2(z)$$
$$x_1(k+1) = -2x_1(k) + x_2(k) + u(k) \quad \text{— (A)}$$

(3), (5) $\Rightarrow$
$$zX_2(z) = 2U(z) - X_1(z) + X_3(z).$$
$$x_2(k+1) = -x_1(k) + x_3(k) + 2u(k) \quad \text{— (B)}$$

(2), (5) $\Rightarrow$
$$zX_3(z) = U(z) - 0.5X_1(z).$$
$$x_3(k+1) = u(k) - 0.5x_1(k). \quad \text{— (C)}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ -0.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u(k).$$

(5) $\Rightarrow$
$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Q. Write the state space eqn for the system described by 3.22

$$y(k+2) = u(k) + 1.7y(k+1) - 0.72y(k) \quad \text{--- (1)}$$

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Sol: Method-1

Let's assume state ~~equation~~ variables.

$$x_1(k) = y(k).$$

$$x_2(k) = y(k+1). \Rightarrow x_1(k+1). \quad \text{--- (2)}$$

$$\therefore \boxed{x_1(k+1) = x_2(k)} \quad \text{--- (A)}$$

(1), (2) ⇒

$$y(k+2) = x_2(k+1) = u(k) + 1.7x_2(k) - 0.72x_1(k) \quad \text{--- (B)}$$

(A) | (B) ⇒

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

Method-2

$$z^2Y(z) = U(z) + z1.7Y(z) - 0.72Y(z)$$

$$\frac{Y(z)}{U(z)} = \frac{1}{z^2 - 1.7z + 0.72} = \frac{z^{-2}}{1 - 1.7z^{-1} + 0.72z^{-2}}$$

$$\frac{Y(z)}{z^{-2}} = \frac{U(z)}{1 - 1.7z^{-1} + 0.72z^{-2}} = Q(z). \quad \text{--- (1)}$$

$$Y(z) = z^{-2}Q(z) \quad \& \quad U(z) = Q(z) - 1.7z^{-1}Q(z) + 0.72z^{-2}Q(z) \quad \text{--- (2)}$$

$$z^{-2}Q(z) = X_1(z) \Rightarrow$$

$$z^{-1}Q(z) = X_2(z) \Rightarrow zX_2(z) = Q(z). \quad \text{--- (3)}$$

$$\therefore zX_1(z) = z^{-1}Q(z) = X_2(z)$$

$$\therefore x_1(k+1) = x_2(k) \quad \text{--- (A)}$$

(2), (3) ⇒

$$Q(z) = U(z) + 1.7X_2(z) - 0.72X_1(z).$$

$$zX_2(z) = U(z) + 1.7X_2(z) - 0.72X_1(z)$$

$$x_2(k+1) = -0.72x_1(k) + 1.7x_2(k) + u(k) \quad \text{--- (B)}$$

(A) | (B) ⇒

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$Y(z) = z^{-2}Q(z) = X_1(z)$$

$$\therefore y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

Exercise

① Write the following transfer function in O.C.F, C.C.F, D.C.F

$$G(z) = \frac{z+3}{(z+1)(z+2)}.$$

Sol

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 3 & 1 \end{bmatrix} x(k)$$

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Pro Given. $G(z) = \dfrac{z+3}{(z+1)(z+2)}$ find state transition matrix.

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Sol $G(z) = \dfrac{z+3}{z^2+3z+2}$ $a_2 = 2$ $a_1 = 3$. $b_1 = 1$ $b_2 = 3$.

$\therefore$ $X(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$

$y(k) = \begin{bmatrix} 3 & 1 \end{bmatrix} X(k).$

$G = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. $H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 1 \end{bmatrix}$.

$\therefore$ state transition matrix $\phi(k) = \text{I.Z.T}\left[z[zI-G]^{-1} \right]$.

$[zI-G] = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} z & -1 \\ 2 & z+3 \end{bmatrix}$

$z[zI-G]^{-1} = \dfrac{z}{z(z+3)+2} \begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix} = \dfrac{1}{(z+2)(z+1)} \begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix}$

$$= \begin{bmatrix} \dfrac{z(z+3)}{(z+1)(z+2)} & \dfrac{z}{(z+1)(z+2)} \\ \dfrac{-2z}{(z+1)(z+2)} & \dfrac{z^2}{(z+1)(z+2)} \end{bmatrix}$$

$$z[zI-G]^{-1} = \begin{bmatrix} \dfrac{2z}{z+1} - \dfrac{z}{z+2} & \dfrac{z}{z+1} - \dfrac{z}{z+2} \\ \dfrac{-2z}{z+1} + \dfrac{2z}{z+2} & \dfrac{-z}{z+1} + \dfrac{2z}{z+2} \end{bmatrix}$$

$\phi(k) = G^k = \text{I.Z.T}\{z[zI-G]^{-1}\}$.

$$\phi(k) = \begin{bmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k & -(-1)^k + 2(-2)^k \end{bmatrix}$$

P20 find $\phi(k)$ of $X(k+1) = G X(k) + H U(k)$

$y(k) = C X(k)$

$G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}$ $H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $C = [1 \ 0]$. and find

State $x(k)$ and $y(k)$ when the $u(k) = 1$ for $k = 0, 1, 2$ and $X(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Sol

$\phi(k) = I.Z.T \left[z (zI - G)^{-1} \right]$

$zI - G = \begin{bmatrix} z & -1 \\ 0.16 & z+1 \end{bmatrix} \Rightarrow [zI - G]^{-1} = \frac{1}{z^2 + z + 0.16} \begin{bmatrix} z+1 & 1 \\ -0.16 & z \end{bmatrix}$

$$z[zI-G]^{-1} = \begin{bmatrix} \frac{z(z+1)}{(z+0.2)(z+0.8)} & \frac{z}{(z+0.2)(z+0.8)} \\ \frac{-0.16z}{(z+0.2)(z+0.8)} & \frac{z^2}{(z+0.2)(z+0.8)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(4/3)z}{(z+0.2)} - \frac{(1/3)}{(z+0.8)} & \frac{5/3\, z}{z+0.2} + \frac{(-5/3)z}{z+0.8} \\ \frac{-0.8/3\, z}{z+0.2} + \frac{0.8/3}{z+0.8} & \frac{-1/3}{z+0.2} + \frac{4/3}{z+0.8} \end{bmatrix}$$

$$\phi(k) = I.Z.T \left[z (zI - G)^{-1} \right] = \begin{bmatrix} 4/3(-0.2)^k - 1/3(-0.8)^k & 5/3(-0.2)^k - \frac{5}{3}(-0.8)^k \\ -\frac{0.8}{3}(-0.2)^k + \frac{0.8}{3}(-0.8)^k & -\frac{1}{3}(-0.2)^k + \frac{4}{3}(-0.8)^k \end{bmatrix}$$

$X(z) = (zI - G)^{-1} z X(0) + (zI - G)^{-1} H U(z)$

$X(z) = (zI - G)^{-1} [z X(0) + H U(z)]$ $U(z) = \frac{z}{z-1}$.

$$zX(0) + HU(z) = \begin{bmatrix} \frac{z^2}{z-1} \\ \frac{-z^2+2z}{z-1} \end{bmatrix}.$$

$$\therefore X(z) = (zI - G)^{-1} [z X(0) + H U(z)] = \begin{bmatrix} \frac{z^2+2z}{(z+0.2)(z+0.8)(z-1)} \\ \frac{(-z^2+1.84z)z}{(z+0.2)(z+0.8)(z-1)} \end{bmatrix}$$

$$x(k) = IZT[X(z)] = \begin{bmatrix} -17/6(-0.2)^k + \frac{22}{9}(-0.8)^k + 25/18 \\ 3.4/6(-0.2)^k - \frac{17.6}{9}(-0.8)^k + 7/18 \end{bmatrix}$$

$\boxed{y(k) = C X(k)}$.

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⇒ Discretization of Continuous - time state - space equation :- (5.25)

$$\dot{x} = Ax + Bu \qquad \dot{x} = \frac{dx}{dt} \qquad (55)$$
$$Y = Cx + Du. \qquad \text{— (A)}$$

$$\dot{x} - Ax = Bu$$

$$e^{-At}\dot{x} - Axe^{-At} = e^{-At}Bu$$

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}Bu.$$

integrating.

$$e^{-At}x(t) = x(0) + \int_0^t e^{-A\tau} Bu(\tau)d\tau.$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau. \quad \text{— (2A)}$$

if the initial state at $t = t_0$ is known

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau)d\tau. \quad \text{— (2B)}$$

The discrete equivalent of equation (A) is.

$$x((k+1)T) = G(T)\,x(kT) + H(T)\,u(kT). \quad \text{— (B)}$$

if we assume the i/p $u(t)$ is unchanged between two consecutive sampling instants, then

$$u(t) = u(kT) \text{ for } kT \le t < kT+T.$$

$$\therefore (2B) \Rightarrow \; x((k+1)T) = e^{A((k+1)T - kT)}x(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\tau)} Bu(\tau)d\tau.$$

$$x((k+1)T) = e^{AT}x(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\tau)} B\,d\tau.\ u(kT)$$

Note

(1) When the system state-space form is in Diagonal or Jordon form then we can directly investigate the controllability & observability.

Observability

Diagonal form — Jordon form

(1). None of the colomns of $m \times n$ matrix 'C', consists of all zero elements.

if $\tau = kT+T-\lambda \Rightarrow d\tau = -d\tau$.

if $\lambda = kT+T-\tau$. if $\tau =$.

$$x((k+1)T) = e^{AT} x(kT) + \int_T^0 e^{A\lambda} B(-d\lambda)\, u(kT)$$

$$x((k+1)T) = e^{AT} x(kT) + \int_0^T e^{A\lambda} B\, d\lambda\; u(kT)$$ (C)

Comparing (B) & (C)

$$G(T) = e^{AT} \qquad H(T) = \int_0^T e^{A\lambda} B\, d\lambda.$$

Ex obtain the discrete state space model of the system given

$$\ddot{y} + 2\dot{y} = u(t).$$

Sol let $x_1(t) = y(t)$ $\quad x_2(t) = \dot{y}(t) = \dot{x}_1(t)$.

$\therefore \dot{x}_2(t) = -2\dot{y}(t) + u(t) = -2x_2(t) + u(t)$

$$\therefore \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$G = e^{AT} = L^{-1}[sI - A]^{-1}.$$

$$e^{AT} = L^{-1}\left\{\begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1}\right\} = L^{-1}\begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

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$$e^{AT} = \begin{bmatrix} 1 & 0.5 - 0.5e^{-2T} \\ 0 & e^{-2T} \end{bmatrix}$$

$$B = \int_0^T \begin{bmatrix} 1 & 0.5 - 0.5e^{-2\lambda} \\ 0 & e^{-2\lambda} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} d\lambda.$$

$$= \int_0^T \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-2\lambda} \\ e^{-2\lambda} \end{bmatrix} d\lambda.$$

Calculation of state transition matrix :-

methods :-

1. z-transform approach – $Z^{-1}[(zI-G)^{-1}z]$.
2. Similarity Transformation.
3. caley - Hamilton Theorem

(3) Caley - Hamilton Theorem :-

" every square matrix satisfies its own characteristic equation".

→ we have G of $n \times n$ square matrix.

The characteristic equation.

$$\Delta(\lambda) \Rightarrow \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n = 0 \quad -(1)$$

acc to C-H.

$$\Delta(G) = G^n + \alpha_1 G^{n-1} + \cdots + \alpha_{n-1} G + \alpha_n I = 0. \quad -(2)$$

Consider a matrix function

$$f(G) = a_0 I + a_1 G + a_2 G^2 + \cdots + a_n G^n + a_{n+1} G^{n+1} + \cdots$$

the degree of the $f(G)$ > order of G.

The Corresponding Scalar polynomial

$$f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n + a_{n+1}\lambda^{n+1} + \cdots$$

$$\frac{f(\lambda)}{\Delta(\lambda)} = q(\lambda) + \frac{g(\lambda)}{\Delta(\lambda)} \quad \text{(3)}$$

← remainder polynomial.

let $g(\lambda) = \beta_0 + \beta_1\lambda + \cdots + \beta_{n-1}\lambda^{n-1}$.

(3) ⇒ $f(\lambda) = q(\lambda)\,\Delta(\lambda) + g(\lambda)$. — (4)

→ if λ_i $i = 1,2,3 \ldots n$ are distinct eigen values of G

then. (4) ⇒ $f(\lambda_i) = q(\lambda_i)\,\Delta(\lambda_i) + g(\lambda_i)$

but $\Delta(\lambda_i) = 0$ ($\because$ in C.E if we substitute λ_i it will be zero)

$\therefore$ $f(\lambda_i) = g(\lambda_i)$ $i = 1,2,3 \ldots n$. — (5).

$\beta_0, \beta_1, \beta_2 \ldots \beta_{n-1}$ are obtained by substituting $\lambda_1, \lambda_2, \ldots \lambda_n$ resly in (5).

→ if G has eigen value λ_j of multiplicity of order m_j then.

$$\left.\frac{d^p}{d\lambda^p}\Delta(\lambda)\right|_{\lambda=\lambda_j} = 0 \qquad p = 0, 1, \ldots (m_j - 1).$$

$\therefore$ &

$$\left.\frac{d^p}{d\lambda^p} f(\lambda)\right|_{\lambda=\lambda_j} = \left.\frac{d^p}{d\lambda^p} g(\lambda)\right|_{\lambda=\lambda_j} \qquad p = 0, 1, 2 \ldots m_j - 1.$$

Eg eigen values are 3, 3, 3, 2, 2, −1. then.

$\lambda_1 = 3$ $m_1 = 3$ $\lambda_3 = -1$.

$j = 1, 2$ p values 0, 1. $\lambda_2 = 2$. $m_2 = 2$

→ in (4) if we substitute G instead of λ acc to Caley-Hamilton.

$$f(G) = q(G)\,\Delta(G) + g(G) \quad \& \quad \Delta(G) = 0.$$

$$\therefore f(G) = g(G) = \beta_0 I + \beta_1 G + \beta_2 G^2 + \cdots + \beta_{n-1} G^{n-1}$$

(3.27)

(57)

~~#~~ if G is of order n, for a scalar polynomial $f(\lambda)$, that corresponds to the given matrix polynomial $f(G)$, we can construct a polynomial $g(\lambda)$ of degree $n-1$. of the form $\beta_0 + \beta_1\lambda + \dots \beta_{n-1}\lambda^{n-1}$

→ equating $f(\lambda)$ and $g(\lambda)$ at the distinct eigenvalues of G from

$f(\lambda_i) = g(\lambda_i)$ $i = 1, 2 \dots n$. and

equating derivatives of $f(\lambda)$ and $g(\lambda)$ at the repeated eigen values

from $\left.\dfrac{d^p}{d\lambda^p} f(\lambda)\right|_{\lambda=\lambda_j} = \left.\dfrac{d^p}{d\lambda^p} g(\lambda)\right|_{\lambda=\lambda_j}$ $p = 0, 1, 2, \dots m_j - 1$.

gives 'n' no. of algebraic equations from which $\beta_0, \beta_1 \dots \beta_{n-1}$ are found.

Q

$$G = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\Delta(\lambda) \Rightarrow |\lambda I - G| = \begin{vmatrix} \lambda & 0 & 2 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-3 \end{vmatrix} = (\lambda-1)^2(\lambda-2) = 0$$

$\therefore \lambda_1 = 1$ of multiplicity 2.

$\lambda_2 = 2$.

$f(\lambda) = G^k$; $g(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$.

$f(\lambda_1) = g(\lambda_1)$.

$\dfrac{df(\lambda_1)}{d\lambda} = \dfrac{d}{d\lambda} f(\lambda_1)$.

$f(\lambda_2) = g(\lambda_2)$

$$\left.\begin{aligned} (1)^k &= \beta_0 + \beta_1 + \beta_2 \; ; \lambda_1 = 1 \\ k(1)^{k-1} &= \beta_1 + 2\beta_2 \; ; \lambda_1 = 1 \\ 2^k &= \beta_0 + 2\beta_1 + 4\beta_2 \; ; \lambda_2 = 2 \end{aligned}\right\} \leftarrow \text{solving}$$

solving. $\beta_0 = 2^k - 2k(1)^{k-1}$.

$\beta_1 = 2[(1)^k - 2^k] + 3k(1)^{k-1}$.

$\beta_2 = (2)^k - (1)^k - k(1)^{k-1}$

$.1^{-1} = k$

$G^k = \beta_0 I + \beta_1 G + \beta_2 G^2$.

Q 2. $x(k+1) = \begin{bmatrix} 0 & 1 \\ -0.21 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-1)^k$. $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$y(k) = x_2(k)$. find S.T.M using ① Caley-Hamilton ② z-transform

Sol. ① $G = \begin{bmatrix} 0 & 1 \\ -0.21 & -1 \end{bmatrix}$.

$|\lambda I - G| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 0.21 & \lambda+1 \end{vmatrix} = 0$ $\quad \lambda_1 = -0.3$, $\lambda_2 = -0.7$

$\therefore \quad f(\lambda) = \lambda^k$. $\qquad g(\lambda) = \beta_0 + \beta_1 \lambda$.

$f(\lambda_1) = g(\lambda_1)$

$f(\lambda_2) = g(\lambda_2)$

$$\left.\begin{aligned} (-0.3)^k &= \beta_0 + \beta_1(-0.3) \\ (-0.7)^k &= \beta_0 + \beta_1(-0.7) \end{aligned}\right\} \Rightarrow \begin{aligned} \beta_0 &= -1.75(-0.3)^k - 0.75(-0.7)^k \\ \beta_1 &= 2.5(-0.3)^k - 2.5(-0.7)^k \end{aligned}$$

$\therefore \quad f(G) = G^k = \beta_0 I + \beta_1 G$

$$G^{(k)} = \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}$$

② I.Z.T $\left[(zI - G)^{-1} z\right]$.

$$z \cdot [zI - G]^{-1} = \begin{bmatrix} \frac{z(z+1)}{(z+0.3)(z+0.7)} & \frac{z}{(z+0.3)(z+0.7)} \\ \frac{-0.21z}{(z+0.3)(z+0.7)} & \frac{z^2}{(z+0.3)(z+0.7)} \end{bmatrix}$$

$$\phi(k) = \text{I Z.T}\left[z(zI - G)^{-1}\right] = \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}$$