The Z-transform (Chapters 2 & 3)

③ ⑫

⇒ **Motivation for using z-transform :-**

Control system Design

Conventional (or) classical control techniques. (uses transform theory) (or state space representation)

modern control techniques
& optimal control
Robust control
Model predictive control
(uses only state space and probability theory).

Why z-transform over Laplace :-

Let the output of an ideal sampler be denoted by $f^*(t)$.

$$L[f^*(t)] = F^*(s)$$

$$F^*(s) = \sum_{k=0}^{\infty} f(kT)\, e^{-kTs} \qquad \text{—— ①}$$

→ since $F^*(s)$ contains the term e^{-kTs}, it is not a rational function of s. when terms with e^{-Ts} appear in a transfer function other than a multiplying factor difficulties arise while taking the inverse Laplace. It is desirable to transfer the irrational function $F^*(s)$ to a rational function by the following.

$$\text{Let} \quad e^{Ts} = z$$

$$\Rightarrow \quad s = \frac{1}{T} \ln z.$$

$$\text{if} \quad s = \sigma + j\omega$$

$$Re[z] = e^{T\sigma} \cos\omega T$$

$$Im[z] = e^{T\sigma} \sin\omega T.$$

① ⇒ $F^{*}(s)\Big|_{s=\frac{1}{T}\ln z} = F(z)$

$$\boxed{F(z) = \sum_{k=0}^{\infty} f(kT)\, z^{-k}.}$$

where $F(z)$ is the z-transform of $f(t)$ at the sampling instants k.

Note if $f(t)$ is Laplace transformable then it also has z-transform.

⇒ **Definition :-** 1) (bilateral z-transform)

For a general Discrete-time signal $x[K]$, the z-transform $X(z)$ is defined as

$$\boxed{X(z) = \sum_{K=-\infty}^{\infty} x[K]\, z^{-K}.}$$

← (bilateral (or) two-sided z-transform).

Note $X(k)$ and $X(kT)$ carries same meaning.

2) **unilateral z-transform :-** (one-sided).

$$X(z) = \sum_{K=0}^{\infty} x[k].\, z^{-k}.$$

⇒ **Region of Convergence:-** (R.O.C)

The Range of values of the Complex variable z for which the z-transform Converges is Called the region of Convergence.

→ For bilateral z-transforms, the ROC distinguishes the same z-transform expression for two signals.

→ ROC is not a matter for causal signals (or) unilateral z-transforms

Example ① find X(z) of $x[k] = a^k u(k)$; a being real. ④ ⑬

and define its ROC.

$$X(z) = \sum_{k=-\infty}^{\infty} a^k u(k) z^{-k}.$$

$$= \sum_{k=-0}^{\infty} (a z^{-1})^k$$

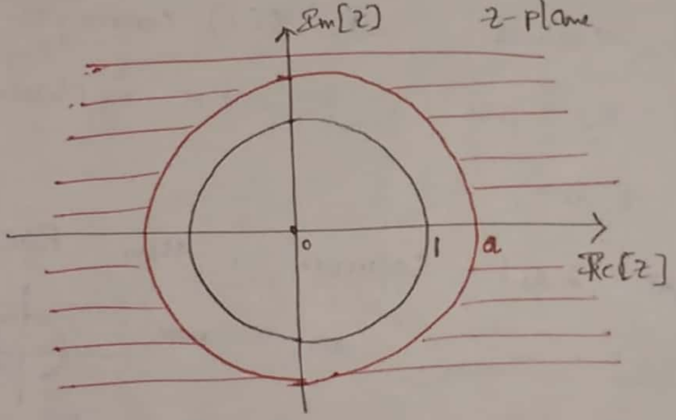
$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \quad — \text{Ⓐ}$$

X(z) converges if and only if $|az^{-1}| < 1$ (or)

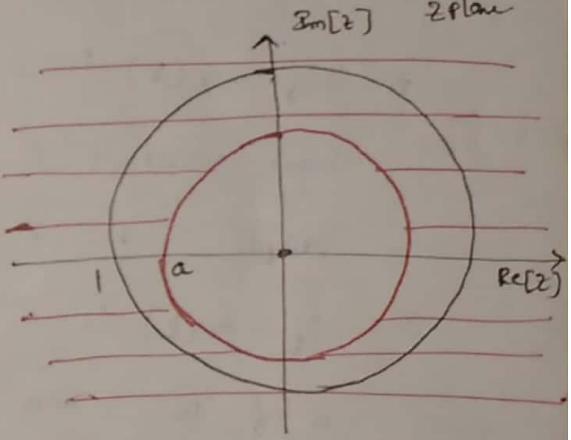
$|z| > |a|$.

→ As z is a complex variable then $|z| = |a|$ is the equation of a circle. of radius 'a'.

$|a| > 1$



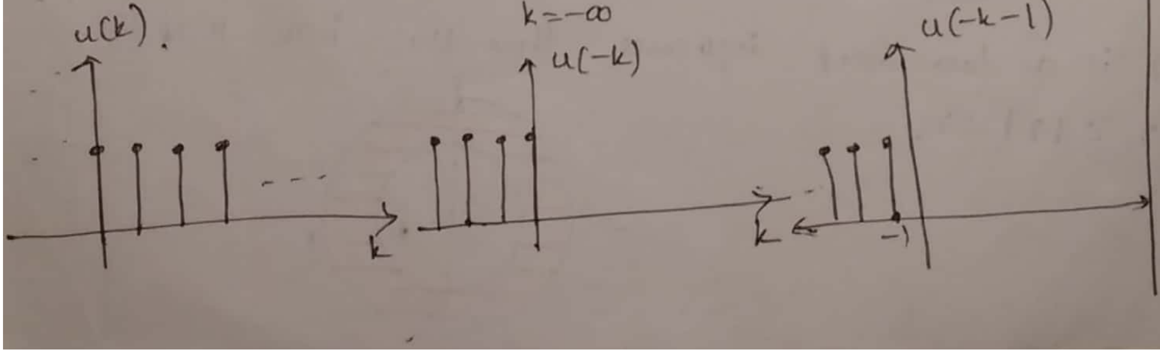
$|a| < 1$

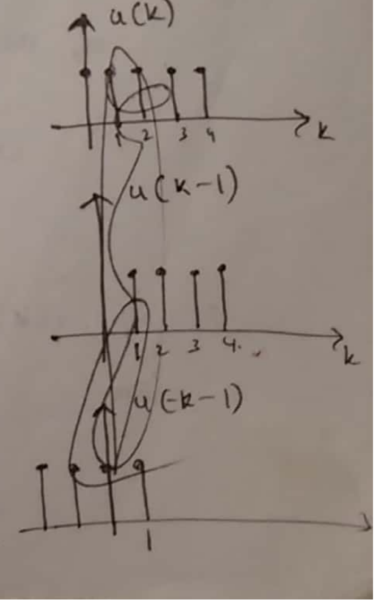


Example 2 Find X(z) of $x[k] = -a^k u(-k-1)$.

$$X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

$$= \sum_{k=-\infty}^{-1} -a^k z^{-k}.$$





$$= -\sum_{k=-\infty}^{-1} (az^{-1})^{k} + 1 - 1$$

$$= \left[1 - \sum_{k=0}^{\infty} (a^{-1}z)^{k}\right]$$

$$= \left[1 - \frac{1}{1 - a^{-1}z}\right] \leftarrow \quad |a^{-1}z| < 1.$$

ROC: $|z| < |a|$

$$= 1 - \frac{a}{a-z}$$

$$= -\frac{z}{a-z}$$

$$X(z) = \frac{z}{z-a} \quad \text{— (B)} \qquad \text{ROC: } |z| < |a|.$$

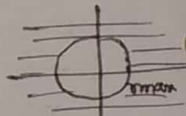
from (A) & (B) it is observed the ROC is different for a ~~z-tran~~ same z-transform. of different signals.

Properties of ROC :- if $X(z)$ is a rational function of z then.

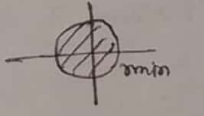
1. The ROC does not contain any poles.

2. if $x(k)$ is a finite sequence and $X(z)$ converges for some value of z then the ROC is the entire z-plane except at $z=0$ or $z=\infty$.

3. if $x(k)$ is a right-sided sequence. then ROC is of the form $|z| > r_{max}$.



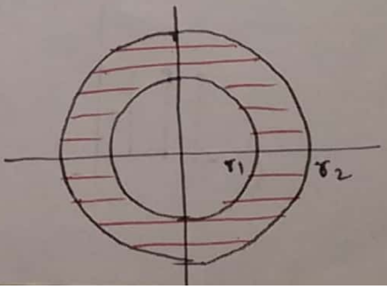
4. if $x(k)$ is a left-sided sequence then the ROC is of the form $|z| < r_{min}$.



where r_{max} = Largest magnitude of poles of $X(z)$.

r_{min} = Smallest magnitude of ~~any~~ poles of $X(z)$

5. If $x(k)$ is a two-sided sequence then the ROC is of the form $r_1 < |z| < r_2$

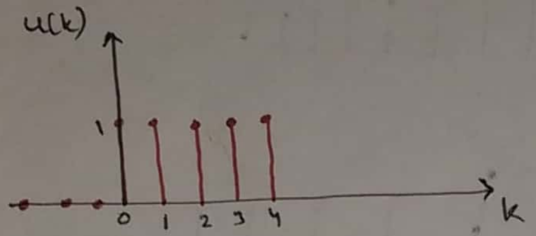


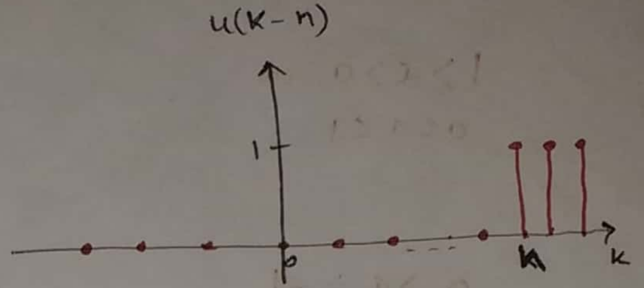
Basic Discreet - signals :-

1. unit - step sequence :- u(k).

$$u(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$u(k-n) = \begin{cases} 1 & k \geq n \\ 0 & k < n. \end{cases}$$

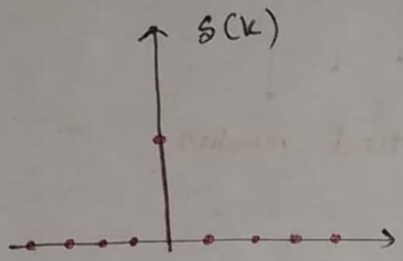


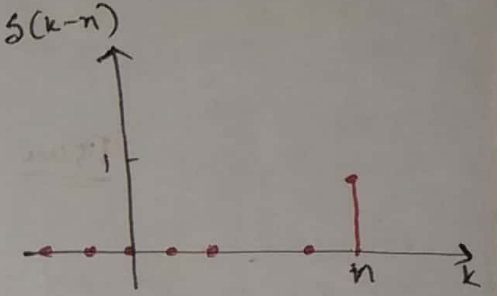


2. unit impulse sequence :-

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$$\delta(k-n) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$





Properties of impulse function :-

1. $x(k)\,\delta(k) = x(0)$
2. $x(k)\,\delta(k-n) = x(n)$
3. $\delta(k) = u(k) - u(k-1)$
4. $u(k) = \sum_{n=-\infty}^{\infty} \delta(k)$

~~3. Exponential Sequences :-~~

~~$x(k) = C\alpha^{k}. \quad \forall k.$~~

~~where C and α are in general Complex numbers.~~

~~can purely real or imaginary numbers~~

⇒ Properties of z - transform :-

→ Most of the real time signals in plants or processes are Continuous signals which are to be discretised to process further in a typical Digital controller. Let $x(t)$ be the Continuous signal, after discretisation by sampling with a period T it becomes $x(kT)$ (or simply $x(k)$).

Note → while listing the properties of z - transform wherever if $x(t) \longrightarrow X(z)$ is found, that means the signal $x(t)$ is discretised afterwards z - transformed.

① Linearity property :-

$$\boxed{\text{if} \quad \begin{array}{l} x(kT) \longrightarrow X(z) \\ y(kT) \longrightarrow Y(z). \end{array}} \qquad (x(kT) = 0; \ k<0$$

then $$\boxed{a_1 x(kT) + b_1 y(kT) \longrightarrow a_1 X(z) + b_1 Y(z)}$$

Proof :- $$a_1\left(z\left[x(kT)\right]\right) + b_1\left(z\left[y(kT)\right]\right) = a_1 X(z) + b_1 Y(z).$$

② Multiplication by a^k :-

$$\left[x(k) = 0 \ ; \ k<0 \right]$$

$$\boxed{\begin{array}{ll} \text{if} & x(k) \longrightarrow X(z) \\ \text{then} & z\left[a^k x(k)\right] := X(a^{-1}z). \end{array}}$$

Proof :-

$$z\left[a^k x(k)\right] = \sum_{k=0}^{\infty} a^k x(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} x(k)\left(a^{-1}z\right)^{-k}.$$

$$= X(a^{-1}z)$$

③ shifting theorem :- a) (Real translation Theorem) :-

$$\text{if} \quad \boxed{\begin{array}{lll} x(t) \longrightarrow & X(z) & \text{then} \\ x(t-nT) \longrightarrow & z^{-n} X(z) & (\text{time - delay}) \\ x(t+nT) \longrightarrow & z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k}\right] & \end{array}}$$

n is a positive number.

Proof

$$z\left[x(t-nT)\right] = \sum_{k=0}^{\infty} x(kT-nT)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} x(kT-nT)\, z^{-k} \cdot \frac{z^{-n}}{z^{-n}}$$

$$= z^{-n}\left[\sum_{k=0}^{\infty} x(kT-nT)\, z^{-(k-n)} + \sum_{k=0}^{n-1} \cancel{x(kT-nT)z^{-k}} - \sum_{k=0}^{n} \cancel{x(kT-nT)z^{-k}}\right]$$

Let $m = k-n$

$$z\left[x(t-nT)\right] = z^{-n} \sum_{m=-n}^{\infty} x(mT)\, z^{-m}.$$

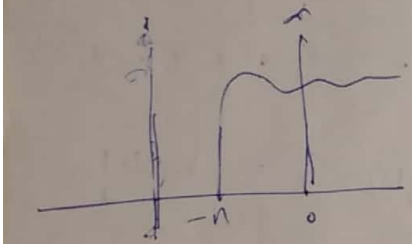
Since $x(mT) = 0$ for $m<0$ ie $\underline{\underline{t<n}}$

$$= z^{-n} \sum_{m=0}^{\infty} x(mT)\, z^{-m}$$

$$\boxed{z\left[x(t-nT)\right] = z^{-n} \cdot X(z).}$$

Ily

$$z\left[x(t+nT)\right] = \sum_{k=0}^{\infty} x(kT+nT)\, z^{-k}.$$



$k+n = m$
$k=0 \Rightarrow m=n$
$k=\infty \Rightarrow m=\infty$

$$\Rightarrow z^{n}\left[\sum_{m=n}^{\infty} x(mT)\, z^{-m} + \sum_{m=0}^{n-1} x(mT)\, z^{-k} - \sum_{m=0}^{n} x(mT)\, z^{-kn}\right]$$

$$= z^{n} \sum_{k=0}^{\infty} x(kT+nT)\, z^{-(k+n)}$$

$$= z^{n}\left[\sum_{k=0}^{\infty} x(kT+nT)\, z^{-(k+n)} + \sum_{k=0}^{n-1} x(kT)\, z^{-k} - \sum_{k=0}^{n-1} x(kT)\, z^{-k}\right]$$

$$= z^{n}\left[\sum_{k=0}^{\infty} x(kT)\, z^{-k} - \sum_{k=0}^{n-1} x(kT)\, z^{-k}\right]$$

$$= z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT)\, z^{-k}\right] /$$

i.e

$$z\left[x(k+1)\right] = zX(z) - zx(0)$$

$$z\left[x(k+2)\right] = z \cdot Z\left[x(k+1)\right] - zx(1) = z^{2}X(z) - z^{2}x(0) - zx(1)$$

$\vdots$

$$z\left[x(k+n)\right] = z^{n}X(z) - z^{n}x(0) - z^{n-1}x(1) - z^{n-2}x(2) - \cdots - zx(n-1).$$

$n =$ positive number.

⑯ (2a)5

*** P20

z transform of function of functions :- (time accumulation).

hd Let $y(k) = \sum_{h=0}^{k} x(h)$ $k = 0, 1, 2, \dots$ ⑦

$y(k)$ is a function of functions $x(h)$

$$y(0) = x(0)$$
$$y(1) = x(0) + x(1)$$
$$y(2) = x(0) + x(1) + x(2).$$
$$\vdots$$
$$y(k) = x(0) + x(1) + \dots + x(k-1) + x(k) \quad \text{—①}$$
$$y(k-1) = x(0) + x(1) + \dots + x(k-1). \quad \text{—②}$$

① − ② ⇒ $y(k) - y(k-1) = x(k)$ $k = 0, 1, 2, \dots$

$$z[y(k) - y(k-1)] = z[x(k)]$$
$$Y(z) - z^{-1} Y(z) = X(z).$$
$$\boxed{Y(z) = \frac{1}{1-z^{-1}} X(z)}$$

shifting theorem :-

ⓑ Complex Translation theorem :-

$$\boxed{\begin{array}{l} \text{if } x(t) \longrightarrow X(z) \text{ then} \\ e^{-at} x(t) \longrightarrow X(ze^{aT}) \end{array}}$$

Proof :- $z[e^{-at} x(t)] = \sum_{k=0}^{\infty} x(kT) e^{-akT} z^{-k}$

$$= \sum_{k=0}^{\infty} x(kT)(ze^{aT})^{-k}.$$

$$z[e^{-at} x(t)] = X(ze^{aT})$$

⇒ Problems on shifting theorems :-

Pr ① Find z transform of

a) $x(k) = \delta(k-k_0)$ c) $x(k) = a^{k+1} u(k+1)$

b) $x(k) = u(k-k_0)$

Sol) a) $z[\delta(k)] = \sum_{k=\infty}^{\infty} \delta(k) z^{-k} = \delta(0)\cdot 1 = \underline{1}$

i.e $\boxed{z[\delta(k)] = 1}$

$z[x(k)] = z[\delta(k-k_0)]$

using $z[x(k-k_0)] = z^{-k_0} X(z)$ property

$z[\delta(k-k_0)] = z^{-k_0}\cdot 1 = z^{-k_0}.$

b) $x(k) = u(k-k_0)$

we know $z[u(k)] = X(z) = \frac{z}{z-1}.$

using shifting property

$z[u(k-k_0)] = z^{-k_0}\cdot\frac{z}{z-1}.$

c) $x(k) = a^{k+1}u(k+1).$

we know $z[a^k u(k)] = \frac{z}{z-a} = X(z)$

$z[a^{k+1}u(k+1)] = z^1\cdot\frac{z}{z-a}$

$= \underline{\frac{z^2}{z-a}}$

Pro ② find z-transform of

ⓐ $e^{-at}\sin\omega t$ ⓑ $e^{bt}\cos\omega t$. ⓒ $te^{-at}.$

Sol) a) $x(t) = e^{-at}\sin\omega t.$

we know $z[\sin\omega t] = \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}$ — ①

⑧ ⑰ (c.c) ④

using the complex translation property

$$z\left[e^{-at}x(t)\right] = X(ze^{aT}).$$

$$\therefore\ z\left[e^{-at}\sin wt\right] = \text{Substitute } z \text{ by } ze^{aT} \text{ in } \cancel{x(z)}\ ①.$$

$$= \frac{ze^{aT}\sin wT}{z^2e^{2aT} - 2e^{aT}\cos wT + 1}$$

b) $z\left[e^{bt}\cos wt\right]$

we know $$z\left[\cos wt\right] = \frac{z^2 - z\cos wT}{z^2 - 2z\cos wT + 1.} \rightarrow ②$$

$$z\left[e^{bt}\cos wt\right] = \text{Substitute } z \text{ by } ze^{-bT} \text{ in } ②$$

$$z\left[e^{bt}\cos wt\right] = \frac{z^2e^{2bT} - ze^{-bT}\cos wT}{z^2e^{2bT} - 2ze^{-bT}\cos wT + 1}$$

ⓒ $z\left[te^{-at}\right]$

we know $$z[t] = z[kT] = \frac{Tz^{-1}}{(1-z^{-1})^2} = X(z).$$

Thus $$z\left[e^{-at}t\right] = \frac{z\,(e^{aT}z)^{-1}}{\left[1-(e^{aT}z)^{-1}\right]^2}$$

$$= \frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2.}$$

Pro find z transform of

$$z\left[\sum_{h=0}^{k-1} x(h)\right] \quad \text{if} \quad y(k) = \sum_{h=0}^{k} x(h).$$

Sol) we know. $$z[y(k)] = \frac{1}{1-z^{-1}}X(z).$$

if $y(k) = \sum_{h=0}^{k} x(h)$ then $\sum_{h=0}^{k-1} x(h)$ is $y(k-1)$.

$$\therefore\ z\left[\sum_{h=0}^{k-1} x(h)\right] = z[y(k-1)] = z^{-1}\cdot X(z)\cdot\frac{1}{1-z^{-1}}$$

④ **Complex Differentiation :-** In the region of Convergence .

if $x(k) \longrightarrow X(z)$ then ROC → R

$z[kx(k)] \longrightarrow -z\frac{d}{dz}X(z)$ ROC → R

$z[k^2x(k)] \longrightarrow \left(-z\frac{d}{dz}\right)^2 X(z)$ ROC → R

$\vdots \qquad \vdots \qquad \vdots$

$z[k^m x(k)] \longrightarrow \left(-z\frac{d}{dz}\right)^m X(z).$ ROC → R

Proof Let $X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}.$

$$\frac{d}{dz}X(z) = \sum_{k=0}^{\infty} (-k)\, x(k)\, z^{-k-1} \text{ ———— }$$

$$= \cancel{-z\sum_{k=0}^{\infty} x(k) k z^{-k}}$$

multiplying both side with z.

$$-z\frac{d}{dz}X(z) = \sum_{k=0}^{\infty} k x(k) z^{-k}. \text{ ——— } ①$$

$$-z\frac{d}{dz}X(z) = z[kx(k)] \text{ ——— } ②$$

differentiating ① w.r.t z.

$$\frac{d}{dz}\left[-z\frac{d}{dz}X(z)\right] = -\sum_{k=0}^{\infty} (k)^2 x(k) z^{-k-1}.$$

multiply both sides ~~by~~ with z

$$\left(z\frac{d}{dz}\right)\left(z\frac{d}{dz}\right)X(z) = \sum_{k=0}^{\infty} k^2 x(k) z^{-k}.$$

$$\boxed{\left(z\frac{d}{dz}\right)^2 X(z) = z[k^2 x(k)].}$$

Illy

$$Z[k^m x(k)] = \left(-z\frac{d}{dz}\right)^m X(z).$$

Problem obtain z-transform of ramp sequence $x(k) = k \; ; \; k \geq 0,1,2 ..$ (Q.7)

from the z-transform of unit step sequence. (9) (18)

Sol we know $z[u(k)] = \frac{1}{1-z^{-1}} = X(z) \quad \left[\frac{z}{z-1} \right]$

then $z[k] = z[k \cdot 1]$ for $k \geq 0$.

$$= -z \frac{d}{dz} X(z)$$

$$= -z \frac{d}{dz} \left(\frac{1}{1-z^{-1}} \right)$$

$$= \frac{z^{-1}}{(1-z^{-1})^2}.$$

⑤ Complex Integration :-

if $z[x(k)] \longrightarrow X(z)$

then $z\left[\frac{x(k)}{k} \right] \longrightarrow \int_z^\infty \frac{X(z_1)}{z_1} dz_1 + \lim_{k \to 0} \frac{x(k)}{k}.$

Proof:- Let $z\left[\frac{x(k)}{k} \right] = G(z) = \sum_{k=0}^{\infty} \frac{x(k)}{k} z^{-k}.$

differentiating w.r.t z

$$\frac{d}{dz} G(z) = -\sum_{k=0}^{\infty} x(k) z^{-k-1}$$

$$= -z^{-1} \sum_{k=0}^{\infty} x(k) z^{-k}$$

$$\frac{d}{dz} G(z) = -z^{-1} X(z)$$

integrating w.r.t z from z to ∞.

$$\int_z^\infty \frac{d}{dz} G(z) \, dz = -\int_z^\infty \frac{X(z_1)}{z_1} dz_1$$

$$G(\infty) - G(z) = -\int_z^\infty \frac{X(z_1)}{z_1} dz_1.$$

$$G(z) = \int_z^\infty \frac{X(z_1)}{z_1} dz_1 + G(\infty)$$

$$G(\infty) = \lim_{z\to\infty} G(z) .$$

by initial value theorem $G(\infty) = g(0)$

$$= \underset{k\to 0}{Lt} \frac{x(k)}{k} .$$

⑥ Partial Differentiation theorem :-

Consider a function $x(t, a)$ or $x(kT, a)$ that is z-transformable. ~~If~~ a is an idependent variable, and let $z[x(t,a)] = z[x(kT, a)] = X(z, a)$. Then

$$\boxed{z\left[\frac{\partial}{\partial a} x(t,a)\right] = z\left[\frac{\partial}{\partial a} x(kT,a)\right] = \frac{\partial}{\partial a} X(z,a) .}$$

Proof:-

$$z\left[\frac{\partial}{\partial a} x(t,a)\right] = \sum_{k=0}^{\infty} \frac{\partial}{\partial a} x(kT, a)\, z^{-k} .$$

$$= \frac{\partial}{\partial a} \sum_{k=0}^{\infty} x(kT, a)\, z^{-k} .$$

$$= \frac{\partial}{\partial a} X(z,a) .$$

Ex find z transform of $t^2 e^{-at}$.

Sol $t^2 e^{-at} = \frac{\partial}{\partial a}(-t e^{-at}) .$

Now ~~if~~ $z[-t e^{-at}] = ?$

$$z[-t] = \frac{-Tz^{-1}}{(1-z^{-1})^2} = X(z)$$

$$z[e^{-at}\cdot(-t)] = X(e^{aT} z)$$

Let $Y(z) = z[-e^{-at} t] = \dfrac{-T e^{-aT} z^{-1}}{(1 - e^{-aT} z^{-1})^2}$

Now if $z[-t e^{-at}] \longrightarrow Y(z)$

$$z\left[\frac{\partial}{\partial a}(-t e^{-at})\right] \longrightarrow \frac{\partial}{\partial a} Y(z,a) .$$

$$\therefore z\left[\frac{\partial}{\partial a}\left(-t e^{-at}\right)\right] = \frac{\partial}{\partial a}\left[\frac{-T e^{-aT} z^{-1}}{(1-e^{-aT}z^{-1})^2}\right] \quad (19)$$

(10)

$$= \frac{T^2 e^{-aT}\left(1+e^{-aT}z^{-1}\right)z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^3}.$$

⑦ ⇒ Real Convolution theorem :-

Let $x_1(t) = 0$, $x_2(t) = 0$ for $t<0$

if $x_1(t) \longrightarrow X_1(z)$

$x_2(t) \longrightarrow X_2(z)$

then $X_1(z)X_2(z) \longrightarrow x_1(kT) * x_2(kT)$.

where $x_1(kT) * x_2(kT) = \sum_{h=0}^{K} x_1(hT)\, x_2(kT-hT)$.

Proof:-

⑧ ⇒ Complex Convolution theorem :-

$z[x_1(k)] \longrightarrow X_1(z)$ $\qquad z[x_2(k)] \longrightarrow X_2(z)$

$$z\left[x_1(k)\, x_2(k)\right] = \frac{1}{2\pi j}\oint_c$$

⑨ ⇒ Initial value theorem :-

if $x(t)$ has z-transform $X(z)$ and if $\lim_{z\to\infty} X(z)$ exists then the initial value of $x(t) = x(0)$ is

$$\boxed{\therefore x(0) = \lim_{z\to\infty} X(z)}$$

Proof:- $X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$

$$\lim_{z\to\infty} X(z) = \underbrace{\lim_{z\to\infty} x(0)}_{\text{No z term.}} + x(1)\cancel{\lim_{z\to\infty} z^{-1}}^{\to 0} + x(2)\cancel{\lim_{z\to\infty} z^{-2}}^{\to 0} + \cdots$$

$$\boxed{\therefore \lim_{z\to\infty} X(z) = x(0)}$$

⑩ **Final - Value Theorem:—**

Let $z[x(t)] \longrightarrow X(z)$

→ ~~and~~ if all the poles of $X(z)(1-z^{-1})$ lie inside the unit circle. with possible exception of a simple pole at $z = 1$.

then final value of $x(t)$ or $x(kT)$ or $x(k)$ is

$$\boxed{\lim_{k\to\infty} x(k) = \lim_{z\to1} (1-z^{-1})\; X(z).}$$

Proof

$$z[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k)\, z^{-k}. \quad — ①$$

$$z[x(k-1)] = z^{-1}X(z) = \sum_{k=0}^{\infty} x(k-1)\, z^{-k} \quad — ②$$

$$① - ② \Rightarrow \sum_{k=0}^{\infty} [x(k) z^{-k}] - \sum_{k=0}^{\infty} [x(k-1)\, z^{-k}] = X(z) - \cancel{z}^{-1} X(z)$$

applying $\underset{z\to1}{\text{Lt}}$

$$\underset{z\to1}{\text{Lt}} \underbrace{\left[\sum_{k=0}^{\infty} x(k) z^{-k} - \sum_{k=0}^{\infty} x(k-1) z^{-k}\right]} = \underset{z\to1}{\text{Lt}} (1-z^{-1}) X(z).$$

$$\downarrow$$

$$[\cancel{x(0)} - \cancel{x(-1)}^{0} + \cancel{x(1)} - \cancel{x(0)} + \cancel{x(2)} - \cancel{x(1)} + \cdots \cancel{} \cdots = x(\infty).$$

$$(\because k<0)]\, z^{-1}$$

$$\underset{z\to1}{\text{Lt}} [x(\infty)\cancel{z^{-\infty}}] = \underset{z\to1}{\text{Lt}} (1-z^{-1})\, X(z).$$

$$\boxed{\underset{k\to\infty}{\text{Lt}}\; x(k) = \underset{z\to1}{\text{Lt}} (1-z^{-1})\, X(z)}$$

Pro. find initial value $x(0)$ of ⑪ ⑳ (2.9)

$$X(z) = \frac{(1-e^{-T})z^{-1}}{(1-z^{-1})(1-e^{-T}z^{-1})}.$$

Sol

$$x(0) = \operatorname*{lt}_{z\to\infty} X(z) = \operatorname*{lt}_{z\to\infty} \frac{(1-e^{-T})z^{-1}}{(1-z^{-1})(1-e^{-T}z^{-1})}$$

$$= 0.$$

Pro find final value $x(\infty)$ of $X(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}$; $a>0$.

Sol

$$x(\infty) = \operatorname*{lt}_{z\to 1} (1-z^{-1})X(z) = \operatorname*{lt}_{z\to 1} (1-z^{-1})\left[\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}\right].$$

$$= \operatorname*{lt}_{z\to 1}\left[1 - \frac{(1-z^{-1})}{1-e^{-aT}z^{-1}}\right]$$

$$= 1 - 0$$

$$= \underline{1}$$

Pro Determine . Initial and final values of following.

1. $X_1(z) = \frac{z}{z^2-3z+2}$ 2. $X_2(z) = \frac{1}{(z-0.1)(z-0.5)(z+0.2)}$

3. $X_3(z) = \frac{z^2(2z-1.5)}{(z-1)(z-0.5)^2}$.

Sol ① $X_1(z) = \frac{z}{z^2-3z+2}$

Final value.

$$x(\infty) = \operatorname*{lt}_{z\to 1} (1-z^{-1})X(z)$$

$$= \operatorname*{lt}_{z\to 1} (1-z^{-1})\frac{z}{(z^2-3z+2)}$$

$$= \operatorname*{lt}_{z\to 1} (1-z^{-1})\frac{z}{(z-2)(z-1)}$$

$$= \operatorname*{lt}_{z\to 1} \frac{1}{z-2}$$

$$x(\infty). = -1.$$

Initial Value

$$\operatorname*{lt}_{z\to\infty} X_1(z) = x_1(0)$$

$$x_1(0) = \operatorname*{lt}_{z\to\infty} \frac{z}{z^2-3z+2}$$

$$= \operatorname*{lt}_{z\to\infty} \frac{1}{z\left(1-\frac{3}{z}+\frac{2}{z^2}\right)}$$

$$x(0). = 0.$$

	Final value	Initial value
② $x_2(z) = \frac{1}{(z-0.1)(z-0.5)(z+0.2)}$	$x_2(\infty) = \underset{z\to 1}{Lt}(1-z^{-1})X_2(z)$ $= \underset{z\to 1}{Lt}\frac{(z-1)}{z}\frac{1}{(z-0.1)(z-0.5)(z+0.2)}$ $= \underline{\underline{0}}$	$x_2(0) = \underset{z\to\infty}{Lt} X_2(z)$ $= \underset{z\to\infty}{Lt}\frac{1}{(z-0.1)(z-0.5)(z+}$ $= \underset{z\to\infty}{Lt}\frac{1}{z^3}\left[\frac{1}{(1-\frac{0.1}{z})(1-\frac{0.5}{z})(1-\frac{0.2}{z})}\right.$ $\underline{\underline{x_2(0) = 0}}$
③ $x_3(z) = \frac{z^2(2z-1.5)}{(z-1)(z-0.5)^2}$	$x_3(\infty) = \underset{z\to 1}{Lt}(1-z^{-1})X_3(z)$ $= \underset{z\to 1}{Lt}\frac{\cancel{(z-1)}}{z}\frac{z^{\cancel{2}}(2z-1.5)}{\cancel{(z-1)}(z-0.5)^2}$ $= \frac{1\cdot(2-1.5)}{(1-0.5)^2} = 2$	$x_3(0) = \underset{z\to\infty}{Lt} x_3(z)$ $= \underset{z\to\infty}{Lt}\frac{z^2(2z-1.5)}{(z-1)(z-0.5)^2}$ $= \underset{z\to\infty}{Lt}\frac{\cancel{z^3}(2-1.5/z)}{\cancel{z^3}(1-1/z)(1-\frac{0.5}{z}}$ $= \frac{2}{1}$ $= 2 \quad x_3(0) = \underline{\underline{2}}$

Summary of Properties :-

Property	Time Domain	z-domain
1. Linearity	$a_1x_1(k) + a_2x_2(k)$	$a_1X(z) + a_2X_2(z)$
2. Time Scaling	$x(\frac{k}{m})$ (Note $Z[x(km)]$ is not allowed because $x(km)$ is irreversible)	$X(z^m)$.
3. Time shifting	$x(k-m)$	$z^{-m}X(z)$.
a) Real Translation	$x(k+m)$	$z^m\left[X(z) - \sum_{k=0}^{m-1} x(k)z^{-k}\right]$.
b) Complex Translation	$e^{-akT}x(kT)$	$X(ze^{aT})$.
4. Time accumulation	$y(k) = \sum_{n=0}^{k} x(n)$	$\frac{1}{1-z^{-1}}X(z)$.
5. Z domain Differentiation	$k^m x(k)$	$\left(-z\frac{d}{dz}\right)^m X(z)$.
6. Initial value	$\underset{t\to 0}{Lt}\, x(t)$	$\underset{z\to\infty}{Lt}\, X(z)$
7. final value	$\underset{t\to\infty}{Lt}\, x(t)$	$\underset{z\to 1}{Lt}(1-z^{-1})X(z)$
[illegible] a^k	$a^k x(k)$	[illegible]

P.2.10

⑫ ㉑

Problems

Pro Find the z-transform of the following.

1. $x(k) = -a^{k} u(-k-1)$

Sol $$X(z) = \sum_{k=-\infty}^{\infty} -a^{k} u(-k-1) z^{-k}$$

$$= -\sum_{k=-\infty}^{-1} (a z^{-1})^{k} + 1 - 1$$

$$= 1 - \sum_{k=0}^{\infty} (a^{-1} z)^{k}$$

$$= 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}}$$

[ROC : $|z| < |a|$.]

2. $x(k) = a^{-k} u(-k-1)$

[∵ $\sum_{k=0}^{\infty} (a^{-1}z)^{k} = \frac{1}{1-a^{-1}z}$

$\sum_{k=0}^{\infty} (az)^{k} = \frac{1}{1-az}$.]

Sol $$X(z) = \sum_{k=-\infty}^{\infty} a^{-k} u(-k-1) z^{-k}.$$

$$= \sum_{k=-\infty}^{-1} a^{-k} z^{-k}.$$

$$= \sum_{k=0}^{\infty} (az)^{k} - 1.$$

$$= \frac{1}{1-az} - 1 = \frac{az}{1-az} = \frac{-z}{z - 1/a}$$

[ROC: $|z| < \frac{1}{|a|}$]

3. $$x(k) = \begin{array}{cccccc} \{5, & 3, & \underset{\uparrow}{-2}, & 0, & 4, & -3\} \\ -2 & -1 & 0 & 1 & 2 & 3 \end{array}$$

Sol $$X(z) = \sum_{k=-2}^{3} x(k) z^{-k}.$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 2z^{2} + 3z - 2 + 4z^{-2} + 3z^{-3}.$$

[ROC $0 < |z| < \infty$.]

$$\Rightarrow \quad x(k) = \begin{cases} a^k & 0 \le k \le N-1 \\ 0 & \text{otherwise.} \end{cases} \qquad a>0.$$

Sol

$$X(z) = \sum_{k=0}^{N-1} a^k z^{-k} = \sum_{k=0}^{N-1} (az^{-1})^k$$

$$= \frac{1-(az^{-1})^N}{1-(az^{-1})}$$

$$X(z) = \frac{1}{z^{N-1}} \left[\frac{z^N - a^N}{z-a} \right]$$

$$\Rightarrow \quad x(k) = \begin{cases} a^k & k \ge 0 \\ 0 & k < 0 \end{cases} \qquad (\text{or}) \quad x(k) = a^k u(k).$$

$$X(z) = \sum_{k=-\infty}^{\infty} a^k u(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} (az^{-1})^k = \frac{1}{1-az^{-1}}$$

$$X(z) = \frac{z}{z-a} \qquad [\text{ROC}: |z| > |a|.]$$

$$\Rightarrow \quad x(k) = \left(\tfrac{1}{2}\right)^k u(k) + \left(\tfrac{1}{3}\right)^k u(k).$$

$x(k) = a^k u(k)$

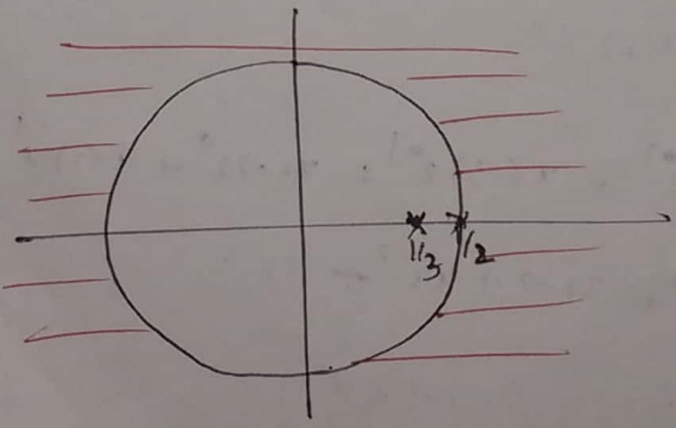
Sol

$$X(z) = \frac{z}{z-1/2} + \frac{z}{z-1/3}.$$

the $X(z) = \dfrac{z}{z-a}.$

ROC

$|z| > 1/2 \qquad |z| > 1/3.$



Prob 10 z - transform of unit ramp function :-

$$x(t) = \begin{cases} t & t \geq 0 \\ 0 & t<0. \end{cases}$$

Sol $x(KT) = KT$ ← discretised signal. ; $T =$ sampling time.

(or) $$x(kT) = \begin{cases} KT & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$X(z) = \sum_{k=0}^{\infty} KT z^{-k}.$$

$$= T\left(z^{-1} + 2z^{-2} + 3z^{-3} + ----\right).$$

$$= T \frac{z^{-1}}{(1-z^{-1})^2}$$

$$= \frac{Tz}{(z-1)^2}$$ ROC: $|z|>1.$

Prob 11

$$x(t) = \begin{cases} e^{-at} & 0 \leq t \\ 0 & t<0. \end{cases}$$

Sol $$x(kT) = \begin{cases} e^{-akT} & k \geq 0 \\ 0 & k<0 \end{cases}$$ $T \rightarrow$ sampling time.

$$X(z) = z\left[e^{-at}\right] = \sum_{k=0}^{\infty} e^{-akT} z^{-k}$$

$$= 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + e^{-3aT} z^{-3} + - -$$

$$= \frac{1}{1 - e^{-aT} z^{-1}}.$$

$$= \frac{z}{z - e^{-aT}}.$$

Q.2.13

⑭ ㉓

Pro $x(t) = \begin{cases} \sin\omega t & t \geq 0 \\ 0 & t<0 \end{cases}$

Sol $\sin\omega t = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$

$\therefore \; x(t) = \begin{cases} \frac{1}{2j}\left[e^{j\omega kT} - e^{-j\omega kT}\right] & k \geq 0 \\ 0 & k<0 \end{cases}$

$$X(z) = \sum_{k=-\infty}^{\infty} x(kT)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2j}\left[e^{j\omega kT} - e^{-j\omega kT}\right] z^{-k}$$

$$= \frac{1}{2j}\left[\sum_{k=0}^{\infty} e^{(j\omega T)k} z^{-k} - \sum_{k=0}^{\infty} e^{-(j\omega T)k} z^{-k}\right]$$

$$= \frac{1}{2j}\left[\frac{z}{z-e^{j\omega T}} - \frac{z}{z-e^{-j\omega T}}\right]$$ Simplifying further

$$X(z) = \frac{z \sin\omega T}{z^2 - 2z\cos\omega T + 1}.$$

Ex $x(t) = \begin{cases} \cos\omega t & t \geq 0 \\ 0 & t<0 \end{cases}$

Sol $x(kT) = \begin{cases} \cos\omega kT & k \geq 0 \\ 0 & k<0 \end{cases}$

$$\cos\omega kT = \frac{e^{j\omega kT} + e^{-j\omega kT}}{2}.$$

$$X(z) = \frac{1}{2}\left[\sum_{k=0}^{\infty} e^{j\omega kT} z^{-k} + \sum_{k=0}^{\infty} e^{-j\omega kT} z^{-k}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{j\omega T}} + \frac{z}{z-e^{-j\omega T}}\right]$$ Simplifying further

$$X(t) = \frac{z^2 - z\cos\omega T}{z^2 - 2z\cos\omega T + 1}.$$

Note Some useful formulas :—

1. $(1-x)^3 = 1-3x+3x^2 - \cdot x^3$
2. $(1-x)^4 = 1-4x+6x^2-4x^3+x^4$
3. $(1-x)^{-1} = 1+x+x^2+x^3+x^4+ \cdots$
4. $(1-x)^{-2} = 1+2x+3x^2+4x^3+5x^4+ \cdots$
5. $(1-x)^{-3} = 1+3x+6x^2+10x^3+15x^4+21x^5+ \cdots$
6. $(1-x)^{-4} = 1+4x+10x^2+20x^3+35x^4+56x^5+84x^6+ \cdots$

⇒ Pulse - transfer function (P.T.F) :—

Let $$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n} \quad (m \le n) \quad — ①$$

(or)

$$X(z) = \frac{b_0 (z-z_1)(z-z_2) \cdots (z-z_m)}{(z-P_1)(z-P_2) \cdots (z-P_n)} \quad — ②$$

→ In control Engineering and Signal Processing $X(z)$ is frequently expressed as a ratio of polynomials of z^{-1}.

$$X(z) = \frac{b_0 z^{-(n-m)} + b_1 z^{-(n-m+1)} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \quad — ③$$

→ ** ① & ② → ① Convienient to find poles and zeros.

② In obtaining z^{-1} by inversion integral method.

→ ~~[illegible]~~ roots of numerator of $X(Z)$ are called zeros

roots of denominator of $X(Z)$ are called poles.

→ ①, ②, ③ are called Pulse transfer functions.

→ ③ is useful in Direct Division method. of finding z^{-1}.

2.13 a

26 17

Comments. Just as in working with the Laplace transformation, a table of z transforms of commonly encountered functions is very useful for solving problems in the field of discrete-time systems. Table 2–1 is such a table.

TABLE 2–1 TABLE OF z TRANSFORMS

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1	—	—	Kronecker delta $\delta_0(k)$ $1, \quad k = 0$ $0, \quad k \neq 0$	1
2	—	—	$\delta_0(n-k)$ $1, \quad n = k$ $0, \quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$

Comments. Just as in worki transforms of commonly encounte in the field of discrete-time system

TABLE 2–1 TABLE OF *z* TRANSFORMS

	$X(s)$	$x(t)$	$x(kT)$ or
1	—	—	Kronecker de 1, k 0, k
2	—	—	$\delta_0(n -$ 1, n 0, n
3.	$\frac{1}{s}$	$1(t)$	$1(k)$
4	$\frac{1}{s+a}$	e^{-at}	e^{-akT}

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2·136

25

TABLE 2–1 (continued)

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
12	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18			a^k	$\frac{1}{1-az^{-1}}$
19			a^{k-1} $k=1,2,3,\ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21			k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22			k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23			k^4a^{k-1}	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24			$a^k\cos k\pi$	$\frac{1}{1+az^{-1}}$
25			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1-z^{-1})^3}$
26		$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}$		$\frac{z^{-m+1}}{(1-z^{-1})^m}$
27			$\frac{k(k-1)}{2!}a^{k-2}$	$\frac{z^{-2}}{(1-az^{-1})^3}$
23	$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}a^{k-m+1}$			$\frac{z^{-m+1}}{(1-az^{-1})^m}$

$x(t) = 0$, for $t < 0$

$x(kT) = x(k) = 0$, for $k < 0$

Unless otherwise noted, $k = 0, 1, 2, 3, \ldots$

Some useful formulas

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

③ ⑮ 2.13 C ㉔

Table 13.1. Unilateral z-transform pairs for several causal DT sequences

DT sequence $x[k] = \frac{1}{2\pi \mathrm{j}} \oint_C X(z) z^{k-1} \mathrm{d}z$	z-transform with ROC $X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$
(1) Unit impulse $x[k] = \delta[k]$	1, ROC: entire z-plane
(2) Delayed unit impulse $x[k] = \delta[k - k_0]$	z^{-k_0}, ROC: entire z-plane, except $z = 0$
(3) Unit step $x[k] = u[k]$	$\frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$, ROC: $\lvert z\rvert > 1$
(4) Exponential $x[k] = \alpha^k u[k]$	$\frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}$, ROC: $\lvert z\rvert > \lvert\alpha\rvert$
(5) Delayed exponential $x[k] = \alpha^{k-1} u[k - 1]$	$\frac{z^{-1}}{1 - \alpha z^{-1}} = \frac{1}{z - \alpha}$, ROC: $\lvert z\rvert > \lvert\alpha\rvert$
(6) Ramp $x[k] = k u[k]$	$\frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z - 1)^2}$, ROC: $\lvert z\rvert > 1$
(7) Time-rising exponential $x[k] = k\alpha^k u[k]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} = \frac{\alpha z}{(z - \alpha)^2}$, ROC: $\lvert z\rvert > \lvert\alpha\rvert$
(8) Causal cosine $x[k] = \cos(\Omega_0 k) u[k]$	$\frac{1 - z^{-1}\cos\Omega_0}{1 - 2z^{-1}\cos\Omega_0 + z^{-2}} = \frac{z[z - \cos\Omega_0]}{z^2 - 2z\cos\Omega_0 + 1}$, ROC: $\lvert z\rvert > 1$
(9) Causal sine $x[k] = \sin(\Omega_0 k) u[k]$	$\frac{z^{-1}\sin\Omega_0}{1 - 2z^{-1}\cos\Omega_0 + z^{-2}} = \frac{z\sin\Omega_0}{z^2 - 2z\cos\Omega_0 + 1}$, ROC: $\lvert z\rvert > 1$
(10) Exponentially modulated cosine $x[k] = \alpha^k \cos(\Omega_0 k) u[k]$	$\frac{1 - \alpha z^{-1}\cos\Omega_0}{1 - 2\alpha z^{-1}\cos\Omega_0 + \alpha^2 z^{-2}} = \frac{z[z - \alpha\cos\Omega_0]}{z^2 - 2\alpha z\cos\Omega_0 + \alpha^2}$, ROC: $\lvert z\rvert > \lvert\alpha\rvert$
(11) Exponentially modulated sine I $x[k] = \alpha^k \sin(\Omega_0 k) u[k]$	$\frac{\alpha z^{-1}\sin\Omega_0}{1 - 2\alpha z^{-1}\cos\Omega_0 + \alpha^2 z^{-2}} = \frac{\alpha z\sin\Omega_0}{z^2 - 2\alpha z\cos\Omega_0 + \alpha^2}$, ROC: $\lvert z\rvert > \alpha$
(12) Exponentially modulated sine II $x[k] = r\alpha^k \sin(\Omega_0 k + \theta) u[k]$, with $\alpha \in R$.	$\frac{A + Bz^{-1}}{1 + 2\gamma z^{-1} + \alpha^2 z^{-2}} = \frac{z(Az + B)}{z^2 + 2\gamma z + \gamma^2}$, ROC: $\lvert z\rvert \le \lvert\alpha\rvert^{(a)}$

(a) Where $r = \sqrt{\frac{A^2\alpha^2 + B^2 - 2AB\gamma}{\alpha^2 - \gamma^2}}$, $\Omega_0 = \cos^{-1}\left(\frac{-\gamma}{\alpha}\right)$, and $\theta = \tan^{-1}\left(\frac{A\sqrt{\alpha^2 - \gamma^2}}{B - A\gamma}\right)$.

DTFT is obtained by computing the z-transform along the unit circle in the complex z-plane.

Table 13.1 lists the z-transforms for several commonly used sequences. Comparing Table 13.1 with Table 11.2, we observe that when the sequence is causal and its DTFT exists, the DTFT can be obtained from the z-transform by substituting $z = \mathrm{e}^{\mathrm{j}\Omega}$. Since the substitution $z = \mathrm{e}^{\mathrm{j}\Omega}$ can only be made if the ROC contains the unit circle, an alternative condition for the existence of the DTFT is the inclusion of the unit circle within the ROC of the z-transform. If the ROC of a z-transform does not include the unit circle, we cannot substitute $z = \mathrm{e}^{\mathrm{j}\Omega}$ and we say that its DTFT cannot be obtained from Eq. (13.8). For example, the ROC of the unit step function is given by $\lvert z\rvert > 1$, which does not contain the

2·13C

Table 13.2. Properties of the z-transform for transform pairs $x[k] \overset{z}{\longleftrightarrow} X(z)$, ROC: R_x; $x[k]u[k] \overset{z}{\longleftrightarrow} X^{(c)}(z)$, ROC: R_x; $x_1[k] \overset{z}{\longleftrightarrow} X_1(z)$, ROC: R_1; $x_2[k] \overset{z}{\longleftrightarrow} X_2(z)$, ROC: R_2

Properties	Time domain	z-domain	ROC
Linearity	$a_1x_1[k] + a_2x_2[k]$	$a_1X_1(z) + a_2X_2(z)$	at least $R_1 \cap R_2$
Time scaling	$x^{(m)}[k]$ for $m = 1, 2, 3, \ldots$	$X(z^m)$	$(R_x)^{1/m}$
Time shifting (non-causal)	$x[k-m]$	$z^m X(z)$	
Time shifting (causal)	$x[k-m]u[k-m]$	$z^m X^{(c)}(z)$	R_x, except for the possible deletion or addition of $z = 0$ or $z = \infty$
	$x[k+m]u[k]$	$z^m X^{(c)}(z) - z^m \sum_{k=0}^{m-1} x[k]z^{-k}$	
	$x[k-m]u[k]$	$z^{-m} X^{(c)}(z) + z^{-m} \sum_{k=1}^{m} x[-k]z^{k}$	
Frequency shifting	$e^{j\Omega_0 k}x[k]$	$X(e^{-j\Omega_0}z)$	R_x
Time differencing	$x[k] - x[k-1]$	$(1 - z^{-1})X(z)$	R_x, except for the possible deletion of the origin
Time accumulation	$y[k] = \sum_{m=0}^{k} x[m]$ [a]	$\frac{z}{z-1}X(z)$	$R_x \cap (\|z\| > 1)$
z-domain differentiation	$kx[k]$	$-z\frac{dX(z)}{dz}$	R_x
Time convolution	$x_1[k] * x_2[k]$	$X_1(z)X_2(z)$	at least $R_1 \cap R_2$
Initial-value theorem		$x[0] = \lim_{z\to\infty} X(z)$	provided $x[k] = 0$ for $k < 0$
Final-value theorem	$x[\infty] = \lim_{k\to\infty} x[k] = \lim_{z\to 1}(z-1)X(z)$		provided $x[\infty]$ exists

[a] Provided that the sequence $y[k]$ has a finite value for all k.

Proof

Using Eq. (13.7), the z-transform of $a_1x_1[k] + a_2x_2[k]$ is calculated as follows:

$$\begin{aligned} Z\{a_1x_1[k] + a_2x_2[k]\} &= \sum_{k=0}^{\infty} \{a_1x_1[k] + a_2x_2[k]\}\, z^{-k} \\ &= a_1 \underbrace{\sum_{k=0}^{\infty} x_1[k]z^{-k}}_{X_1(z)} + a_2 \underbrace{\sum_{k=0}^{\infty} x_2[k]z^{-k}}_{X_2(z)}, \end{aligned}$$

which proves the algebraic expression, Eq. (13.16). To determine the ROC of the linear combination, we note that the z-transform $X_1(z)$ is finite within the specified ROC, R_1. Similarly, $X_2(z)$ is finite within its ROC, R_2. Therefore, the linear combination $a_1X_1(z) + a_2X_2(z)$ should be finite at least within region R

-: Inverse z - transform :-

→ Inverse z-transform z^{-1} yields a unique $x(k)$ but doesnt yield a unique $x(t)$. The exact reconstruction of the signal $x(t)$ from $x(k)$ is dependent on sampling rate.

→ if $X(z)$ is the z transform of $x(k)$

$$x(k) = \frac{1}{2\pi j} \oint_{c} X(z)\, z^{k-1}\, dz.$$

where C is a closed contour traversed in the C.C wise

→ solving the above equation involve the application of contour integration, which is difficult. So it is seldom used.

→ There are the following methods to evaluate z^{-1}.

1. Direct - Division method.
2. Computational method (or) Difference equation approach. not required
3. Partial - fraction method
4. Inversion - integral Method.

(i) Direct - Division method :- (Power series method)

→ $X(z)$ is expanded into an infinite power series in z^{-1}.

→ This method is useful when

i) it is difficult to obtain the closed - form expression

2) it is desired to find only the first several terms of x(k

→ let $X(z) = \sum_{k=0}^{\infty} x(kT)\, z^{-k} = x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots + \dots$

→ then $x(kT)$ is the co-efficient of z^{-k}. So by inspection of the power series we can write the sequence $x(k)$.

steps :-

Step ① if $X(z)$ is rational function then

a) arrange numerator and denominator terms in increasing powers of z^{-1}.

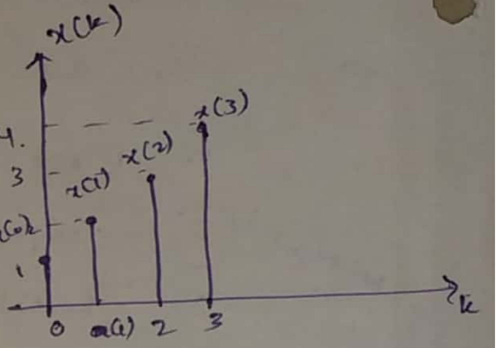
b) then perform Direct division.

Step ② The Direct - division gives an infinite power series in increasing power of z^{-1}. Now Compare the Co-efficients of z^{-k}. $k = 0, 1, 2 - - -$.

Ex Find z^{-1} of the following by Power series Method :

a) $X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

b) $X(z) = \dfrac{10z + 5}{(z-1)(z-0.2)}$



Sol

a) $X(z) = \underset{x(0)}{\underset{\downarrow}{1}} + \underset{x(1)}{\underset{\downarrow}{2z^{-1}}} + \underset{x(2)}{\underset{\downarrow}{3z^{-2}}} + \underset{x(3).}{\underset{\downarrow}{4z^{-3}}}$

b) $X(z) = \dfrac{10z + 5}{(z-1)(z-0.2)} = \dfrac{10z + 5}{z^2 - 1.2z + 0.2} \times \dfrac{z^{-2}}{z^{-2}}$

$$= \frac{10z^{-1} + 5z^{-2}}{1 - 1.2z^{-1} + 0.2z^{-2}}$$

$$1 - 1.2z^{-1} + 0.2z^{-2} \,\big)\, 10z^{-1} + 5z^{-2} \,\big(\, 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + - - -$$

$$\underline{-\;\cancel{10z^{-1}} \mp 12z^{-2} \pm 2z^{-3}.}$$

$$\cancel{17z^{-2}} - 2z^{-3}$$

$$\underline{-\;\cancel{17z^{-2}} \mp 20.4z^{-3} + 3.4z^{-4}.}$$

$$\cancel{18.4z^{-3}} - 3.4z^{-4}$$

$$\underline{-\;\cancel{18.4z^{-3}} \mp 22.08z^{-4} + 3.68z^{-5}}$$

$$18.68z^{-4} - 3.68z^{-5}$$

$$\underline{18.68z^{-4} - 22.41z^{-5} + 3.736z^{-6}}$$

$\therefore X(z) = 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + - - -$

$\therefore$ $x(0) = 0$ $\quad x(2) = 17$

$x(1) = 10$ $\quad x(3) = 18.4$

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2.15

~~Not necessary~~

Method - 2 Difference - equation approach :- ㉘ ⑲

→ let $\frac{Y(z)}{X(z)} = \frac{n(z)}{m(z)} = G(z)$.

$n(z)$, $m(z)$ should be in ~~increasing powers of~~ in terms of z.

→ Select $x(k)$ = Kronecker delta input = $\delta_0(kT)$

$$\delta_0(kT) = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$$

$\boxed{\therefore \cancel{X(z)} = \cancel{1}}$ $\boxed{X(z) = 1}$

→ $m(z)\, Y(z) = n(z)\, X(z)$.

→ The above expression will be in the form of time delay or time advance. By substituting proper values of k we can find the sequence.

Ex

$$G(z) = \frac{0.4673 z^{-1} - 0.3393 z^{-2}}{1 - 1.5377 z^{-1} + 0.6607 z^{-2}} = \frac{Y(z)}{X(z)}.$$

$$(0.4673z - 0.3393)\, X(z) = (z^2 - 1.5327 z + 0.6607)\, Y(z).$$

$$y(k+2) - 1.5327\, y(k+1) + 0.6607\, y(k) = 0.4637\, x(k+1) - 0.3393\, x(k) \quad ①$$

$x(0) = 1$ $\quad x(k) = 0 \quad k \neq 0$. & $y(k) = 0;\ k < 0$.

→ in ① put $k = -2$ to get $y(0)$

$$y(0) - 1.5327\, y(-1) + 0.6607\, y(-2) = 0.4637\, x(-1) - 0.3393\, x(-2)$$

$$\boxed{y(0) = 0}$$

→ in ① put $k = -1$ to get $y(1)$.

$$y(1) - 1.5327\, y(0) + 0.6607\, y(-1) = 0.4637\, x(0) - 0.3393\, x(-1)$$

$$\boxed{y(1) = 0.4673.}$$

→ with the initial data $y(0) = 0$, $y(1) = 0.4673$, $x(0) = 0$ ~~& x(k)=0 for k≠0~~ & $x(k)$ for $k \neq 0$ eqn ① should be solved.

③

Partial fraction method of finding z^{-1} :-

⇒ Partial fraction Decomposition :-

1. The Numerator must be of a lower degree than the denominator. If not, then it must be divided out.

2. The denominator is factorised into its prime factors. These determine the shapes of the partial fractions.

3. A first order factor $(x+a)$ gives a partial fraction $\frac{A}{x+a}$ ... A is constant to be evaluated

4. A repeated factor $(x+a)^2$ or $(x+a)^3$ gives

$$\frac{A}{(x+a)} + \frac{B}{(x+a)^2} \qquad \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3}.$$

A, B, C are to be evaluated.

5. A quadratic factor (x^2+px+q) gives.

$$\frac{Dx+E}{x^2+px+q}$$, D & E are to be evaluated.

6. A repeated quadratic factor $(x^2+px+q)^2$ gives

$$\frac{Dx+E}{x^2+px+q} + \frac{Fx+G}{(x^2+px+q)^2}$$

D, E, F, G are to be evaluated.

Summary

(29) (20)

2.16

① $$\frac{\boxed{}}{(x\pm a)(x\pm b)(x\pm c)--(x\pm e)} = \frac{A}{x\pm a} + \frac{B}{(x\pm a)} + ---+ \frac{E}{(x\pm a)}$$

② $$\frac{\boxed{}}{(x\pm a)^2(x\pm b)} = \frac{A}{(x\pm b)} + \frac{B}{(x\pm a)} + \frac{C}{(x\pm a)^2}$$

③ $$\frac{\boxed{}}{(px+q)^m(x\pm a)} = \frac{A}{(x\pm a)} + \frac{B}{(px+q)} + \frac{C}{(px+q)^2} + ---+ \frac{J}{(px+q)}$$

④ $$\frac{\boxed{}}{ax^2+bx+c} = \frac{Ax+B}{ax^2+bx+c.}$$

⑤ $$\frac{\boxed{}}{(ax^2+bx+c)(x+a)^2} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{Dx+E}{(ax^2+bx+c)}$$

⑥ $$\frac{\boxed{}}{(x+a)(ax^3+bx^2+cx+d)} = \frac{Ax^2+Bx+C}{ax^3+bx^2+cx+d.} + \frac{D}{x+a.}$$

⇒. How to evaluate A, B, C, D --- ~~Coefficients~~ . Constants?

⇒ Application of Partial fraction Decomposition to evaluate z^{-1} transform :-

→ Consider . X(z) as a proper fraction $(m \le n)$.

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + ----+ b_{m-1} z + b_m}{z^n + a_1 z^{n-1} + ---+ a_{n-1} z + a_n.} \quad ; \quad m \le n$$

i.e order of Numerator is less than order of denominator

→ To expand X(z) into partial fractions . we first factorise the denominator polynomial of X(z) and find the poles of X(z).

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + ---+ b_{m-1} z + b_m}{(z-P_1)(z-P_2)+---(z-P_n).}$$

***

→ Now expand $\frac{X(z)}{z}$ into partial fractions so that each term is easily recognisable. ~~in a [illegible] of z-transform.~~

$$\frac{X(z)}{z} = \frac{a_1}{z-P_1} + \frac{a_2}{z-P_2} + \cdots + \frac{a_n}{z-P_n}$$ (real & distinct roots)

where $a_i = \left[(z-P_i)\frac{X(z)}{z}\right]_{z=P_i}$

→ ~~if $\frac{X(z)}{z}$ [illegible] has multiple poles at $z = P_1$~~

~~$\frac{X(z)}{z} = \frac{a_1}{(z-P_1)^r} + \frac{a_2}{(z-P_1)} + \cdots \frac{a_r}{(z-P_1)^r} + \frac{b_1}{(z-P_2)} + \frac{b_2}{(z-P_3)} + \cdots + \frac{b_{n-r}}{(z-P_n)}$.~~

~~($\therefore$ $n = m + r$. total poles = repeater poles + distinct pole)~~

~~where $b_i = \left[(z-P_i)\frac{X(z)}{z}\right]_{z=P_i}$~~

~~$a_i = \left\{\frac{d^{r-1}}{dz}\left[(z-P_i)^r \frac{X(z)}{z}\right]\right\}_{z=P_i}$~~

~~where P_i are the roots of multiplicity.~~

~~Ex: $\frac{X(z)}{z} = \frac{a_1}{z-P_1}$~~

→ if $\frac{X(z)}{z}$ has 'r' multiple poles at $z = P_1$. $(r \leq n.)$

$$\frac{X(z)}{z} = \frac{a_1}{(z-P_1)^r} + \frac{a_2}{(z-P_1)^{r-1}} + \cdots + \frac{a_r}{(z-P_1)} + \cancel{\frac{b_1}{(z-P_2)}} + \frac{b_2}{(z-P_2)} + \cdots$$

then $$a_k = \frac{1}{(k-1)!}\left\{\frac{d^{k-1}}{dz^{k-1}}\left[(z-P_1)^r \frac{X(z)}{z}\right]\right\}_{z=P_1}$$

$k = 1, 2, \cdots r.$

Ex $$\frac{X(z)}{z} = \frac{a_2}{(z-p_1)} + \frac{a_1}{(z-p_1)^2} + \frac{b_1}{(z-p_2)}$$ ㉚ ②.

total roots $n = 3$

repeated roots $r = 2$

distinct roots $n-r = 1.$

$$a_i = \left\{ \frac{d^{r-1}}{dz^{r-1}} \left[(z-p_1)^r \frac{X(z)}{z} \right] \right\}_{z=p_1}$$

$$a_1 = \left\{ (z-p_1)^2 \frac{X(z)}{z} \right\}_{z=p_1}$$

$$a_2 = \left\{ \frac{d}{dz} \left[(z-p_1)^2 \frac{X(z)}{z} \right] \right\}_{z=p_1,}$$

Bo find inverse transform of the following.

① $F(z) = \frac{1}{z(z-0.2)}$ ② $X(z) = \frac{2z^3+z}{(z-2)^2(z-1)}$

③ $X(z) = \frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$ ④ $x(z) = \frac{z^2+z+2}{(z-1)(z^2-z+1)}$

Sol ① $$F(z) = \frac{1}{z(z-0.2)}$$

$$\frac{F(z)}{z} = \frac{1}{z^2(z-0.2)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{(z-0.2)}$$

$$A = \left\{ \cancel{(z)^2} \frac{1}{\cancel{z^2}(z-0.2)} \right\}_{z=0} = -\frac{1}{0.2} = -5$$

$$B = \left[\frac{d}{dz} \left\{ \cancel{z^2} \frac{1}{\cancel{z^2}(z-0.2)} \right\} \right]_{z=0} = \left. \frac{-1}{(z-0.2)^2} \right|_{z=0} = \frac{-1}{0.04.}$$
$$= -25$$

$$C = \left. \cancel{(z-0.2)} \frac{1}{z^2\cancel{(z-0.2)}} \right|_{z=0.2} = \frac{1}{0.04.} = 25$$

$$\therefore \frac{F(z)}{z} = \frac{-5}{z} - \frac{25}{z^2} + \frac{25}{(z-0.2)}$$

$$F(z) = -5 - \frac{25}{z} + \frac{25\,z}{(z-0.2)}$$

$$\boxed{f(k) = -5\,\delta(k) - 25\,\delta(k-1) + 25(0.2)^k u(k).}$$

Sol 2

$$X(z) = \frac{2z^3+z}{(z-2)^2(z-1)}$$

Sol

$$\frac{X(z)}{z} = \frac{2z^2+1}{(z-2)^2(z-1)} = \frac{A}{z-2} + \frac{B}{(z-2)^2} + \frac{C}{z-1}$$

$$A = \left[(\cancel{z-2})^{\cancel{2}} \frac{2z^2+1}{(\cancel{z-2})^2(z-1)}\right]_{z=2} = 5$$

$$B = \left\{\frac{d}{dz}\left[(\cancel{z-2})^2 \frac{2z^2+1}{(\cancel{z-2})^{\cancel{2}}(z-1)}\right]\right\}_{z=2} = \left.\frac{(z-1)(4z) - (2z^2+1)(1)}{(z-1)^2}\right|_{z=2}$$

$$= \frac{4-5}{1} = \underline{\underline{-1}}$$

$$C = \left[(\cancel{z-1}) \frac{2z^2+1}{(z-2)^2(\cancel{z-1})}\right]_{z=1} = \underline{\underline{3}}$$

$$\therefore X(z) = \frac{5z}{z-2} + \frac{(-z)}{(z-2)^2} + \frac{3z}{z-1} \quad \text{apply } \underline{\underline{z^{-1}}}$$

$$x(k) = 5(2)^k u(k) - \frac{1}{2}k\,(2)^k u(k) + 3(1)^k u(k).$$

Sol 4

$$X(z) = \frac{z^2+z+2}{(z-1)(z^2-z+1)} = \frac{z^2+z+2}{(z-1)(z-0.5-1.5j)(z-0.5+1.5j)}$$

$$\frac{X(z)}{z} = \frac{z^2+z+2}{z(z-1)(z-0.5-1.5j)(z-0.5+1.5j)}$$

$$= \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-0.5-1.5j)} + \frac{D}{(z-0.5+1.5j)}$$

$$A = \left|\cancel{z}\left[\frac{z^2+z+2}{\cancel{z}(z-1)(z^2-z+1)}\right]\right|_{z=0} = \underline{\underline{-2}}$$

$$B = \left.\frac{z^2+z+2}{z(z^2-z+1)}\right|_{z=1.} = \underline{\underline{4}}$$

$$C = \left.\frac{z^2+z+2}{z(z-1)(z\cancel{+}0.5+1.5j)}\right|_{z=0.5+1.5j} = -0.4+0.0667j$$

2.18

(31)

$$D = \left.\frac{z^2+z+2}{z(z-1)(z-0.5+1.5j)}\right|_{z=0.5-1.5j} = -0.4-0.067j$$

(22)

Observation :- If roots are complex conjugate

i.e if $\dfrac{X(z)}{z} = \dfrac{A}{(z-\alpha+j\beta)} + \dfrac{B}{(z-\alpha-j\beta)}$

then $\boxed{B = A^*}$

i.e if $A = p+qj$ then

$B = p-qj$

$$\frac{X(z)}{z} = \frac{-2}{z} + \frac{4}{z-1} + \frac{(-0.4+0.667j)}{z-0.5-1.5j} + \frac{-0.4-0.667j}{(z-0.5+1.5j)}$$

$$x(k) = -2\,\delta(k) + 4(1)^k u(k) + (-0.4+0.667j)(0.5+1.5j)^k u(k) + (-0.4-0.667j)(0.5-1.5j)^k u(k)$$

Not a better sequence.

*** **alternate method** :-

$$\frac{X(z)}{z} = \frac{z^2+z+2}{(z-1)(z^2-z+1)}$$

$$X(z) = \frac{A}{z-1} + \frac{Bz+C}{z^2-z+1} = \frac{z^2+z+2}{(z-1)(z^2-z+1)} \quad \text{— ①}$$

$$A = \left.\frac{z^2+z+2}{z^2-z+1}\right|_{z=1} = \underline{\underline{4}}$$

from ①

$$A(z^2-z+1) + (Bz+C)(z-1) = z^2+z+2$$

$$z^2(A+B) + z(-A-B+C) + A-C = z^2+z+2$$

equating co-efficients

$A+B = 1.$

$B = 1-A = -3.$

$A-C = 2$

$C = A-2 = 2.$

$$\therefore \quad X(z) = \frac{4}{z-2} + \frac{-3z+2}{z^2 \cancel{+2}z+\cancel{1}}$$

$$= \frac{4z^{-1}}{\cancel{1}-z^{-1}} + \frac{(-3z^{-1}+2z^{-2})}{1-z^{-1}+z^{-2}} \quad \text{①}$$

we know

$$\left.\begin{aligned} z\left[e^{-\alpha kT}\cos\omega kT\right] &= \frac{1-e^{-aT}z^{-1}\cos\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}} \\ z\left[e^{-\alpha kT}\sin\omega kT\right] &= \frac{e^{-aT}z^{-1}\sin\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}} \end{aligned}\right\} \text{Ⓐ}$$

Comparing the denominators of ① and Ⓐ

$$-2e^{-aT}\cos\omega T = -1. \quad ; \quad e^{-2aT} = 1.$$

$$\therefore \cos\omega T = 1/2 = 0.5.$$

$$\omega T = \pi/3.$$

$$\sin\omega T = \sqrt{3}/2$$

① ⇒

$$X(z) = \frac{4z^{-1}}{1-z^{-1}} - \frac{3z^{-1}(1-0.5z^{-1})}{1-z^{-1}+z^{-2}} + z^{-1}\frac{0.5z^{-1}\times\left(\frac{\sqrt{3}}{2}\right)}{1-z^{-1}+z^{-2}\left(\frac{\sqrt{3}}{2}\right)}$$

$$= 4z^{-1}\cdot\frac{1}{1-z^{-1}} - 3z^{-1}\cdot\left(\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right) + z^{-1}\cdot\frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}/2\,z^{-1}}{1-z^{-1}+z^{-2}}\right)$$

$$x(k) = 4u(k-1) - 3\cos\frac{(k-1)\pi}{3}u(k-1) + \frac{1}{\sqrt{3}}\sin\frac{(k-1)\pi}{3}.$$

It is more convinient form.

④ Inversion Integral Method :-

Residue theorem :- If a function F(z) is analytic within and on a closed curve 'C', except at a finite number of poles $z_1, z_2, ---, z_m$ inside 'C', then the integral of F(z) taken counter clockwise around 'C' is equal to $2\pi j$ times the sum of the residues at poles $z_1, z_2, ---z_m$.

$$\oint_C F(z)\,dz = 2\pi j\,(k_1+k_2+---+k_m) \leftarrow$$ { Cauchy's Residue theorem

where k_i = Residue at pole z_i $\quad i = 1,2---m.$

Inversion integral for the z-transform :-

→ Consider a circle 'c' with its center at the origin of z-plane such that all poles of $X(z)\,z^{k-1}$ are inside it.

Then
$$\boxed{x(kT) = \frac{1}{2\pi j}\oint_c X(z)\,z^{k-1}\,dz.} \quad \text{—①}$$

→ The evaluation of the inversion integral can be done by

$$\oint_C X(z)\,z^{k-1}\,dz = 2\pi j\,(k_1+k_2+---+k_m) \quad \text{—②}$$

where $k_1, k_2, ---k_m$ are the residues of $X(z)z^{k-1}$ at poles $z_1, z_2, ---z_m$ resply.

∴ from ① & ② ⇒ $x(kT) = k_1+k_2+---+k_m.$

→ if the denominator of $X(z)\,z^{k-1}$ contains a simple pole at $z=z_i$ then

$$\boxed{K = \lim_{z\to z_i}\left[(z-z_i)\,X(z)\,z^{k-1}\right]}$$

→ if $X(z)\,z^{k-1}$ contains multiple pole z_j of order q, then the Residue K is given by

$$\boxed{K = \frac{1}{(q-1)!}\lim_{z\to z_j}\frac{d^{q-1}}{dz^{q-1}}\left[(z-z_j)^q\,X(z)\,z^{k-1}\right]}$$

~~Problems~~

Note* choice between Inversion integral Method & Partial fraction method?

→ if $X(z)\, z^{k-1}$ has no poles at the origin $z=0$ then inversion integral method is a simple technique to obtain inverse z- transform.

→ if $X(z)\, z^{k-1}$ has a simple pole (or) multiple pole at $z=0$ then calculations may become cumbersome and the Partial - fraction expansion, is simpler to apply.

Problem Solve for $x(k)$ of the following using ① Inversion integral Method ② Partial fraction method.

$$X(z) = \frac{(1-e^{-aT})\, z}{(z-1)(z-e^{-aT})}.$$

Sol ① ~~X(z)~~ Partial fraction method.

$$\frac{X(z)}{z} = \frac{(1-e^{-aT})}{(z-1)(z-e^{-aT})} = \frac{A}{z-1} + \frac{B}{z-e^{-aT}}$$

$$A = \left.\frac{1-e^{-aT}}{z-e^{-aT}}\right|_{z=1} = 1.$$

$$B = \left.\frac{1-e^{-aT}}{z-1}\right|_{z=e^{-aT}} = -1.$$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-e^{-aT}} = u(k) - e^{-akT};\quad \underline{k=0,1,2\ldots}$$

② Inversion integral method :-

$$X(z)\, z^{k-1} = \frac{z^{k}(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

Poles are at $z=1$, $z=e^{-aT}$

two poles so, two residues will exist

$$\therefore \quad x(k) = k_1 + k_2$$

$k_1 =$ residue at $z = 1$:

$$\lim_{z \to 1} \left[\frac{(z-1)\,(1-e^{-aT})\,z^k}{(z-1)(z-e^{-aT})} \right] = 1.$$

$k_2 =$ residue at $z = 2$

$$\lim_{z \to e^{-aT}} \left[\frac{(z-e^{-aT})\,(1-e^{-aT})\,z^k}{(z-1)\,(z-e^{-aT})} \right] = -e^{-akT}.$$

$$\therefore \quad x(k) = 1 - e^{-akT} \qquad k \geq 0.$$

*** **Note** if $X(z)$ has a zero of order r at the origin then $X(z)\,z^{k-1}$ will involve a zero of order $r+k-1$ at the origin.

→ if $r \geq 1 \Rightarrow r+k-1 \geq 0$ if $k \geq 0$, Z Positive No. So in $X(z)\,z^{k-1}$. there will be no pole at $z=0$

then $x(k) = K_1 + K_2 + \cdots + K_m$. —— ①

→ if $r \leq 0$ then $r+k-1$ depends on k value. (-ve number). for some ~~values of~~ positive values of $k \Rightarrow Z$ i.e poles exist at $z=0$.. Then the above way of finding $x(k)$ is slightly modified.

very very imp. *** **Example** find $x(k)$ from $X(z) = \dfrac{10}{(z-1)(z-2)}$ using inversion integral Method.

Sol:
$$X(z)\,z^{k-1} = \frac{10\,z^{k-1}}{(z-1)\,(z-2)}$$

for $k=0$ $\quad X(z)\,z^{k-1} = \dfrac{10}{z(z-1)(z-2)}$

for $k=1,2\ldots$ $\quad X(z)\,z^{k-1} = \dfrac{10\,z^{k-1}}{(z-1)(z-2}$

K=0

$$X(z) = \frac{10}{z(z-1)(z-2)}$$

$$x(0) = \underbrace{k_1}_{\text{at } z=0} + \underbrace{k_2}_{\text{at } z=1} + \underbrace{k_3}_{\text{at } z=2}$$

$$K_1 = \left.\frac{10}{(z-1)(z-2)}\right|_{z=0} = 5$$

$$K_2 = \left.\frac{10}{z(z-2)}\right|_{z=1} = -10$$

$$K_3 = \left.\frac{10}{z(z-1)}\right|_{z=2} = 5.$$

$$\therefore \; x(0) = 5-10+5 = 0.$$

K>0

$$X(z) = \frac{10}{(z-1)(z-2)} z^{k-1}.$$

$$x(k) = K_1 + K_2$$

$$K_1 = \left.\frac{10\,z^{k-1}}{z-2}\right|_{z=1} = -10$$

$$K_2 = \left.\frac{10\,z^{k-1}}{z-1}\right|_{z=2} = 10\,2^{k-1}.$$

$$\therefore \; x(k) = \begin{cases} 0 & K=0 \\ -10+10\,2^{k-1} & k=1,2,3\ldots \end{cases}$$

2-21

Exercises:-

Pro: obtain the z-transform of

$X(z) = \dfrac{z(z+2)}{(z-1)^2}$ using all ~~the~~ 3 Methods.

i.e 1. Power Series Method

2. Partial expansion method

3. Inversion Integral method.

Sol ① Power Series method :-

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

$$1-2z^{-1}+z^{-2} \overline{\big)\, 1+2z^{-1}} \big(1+4z^{-1}+7z^{-2}+10z^{-3}+ \cdots$$

$$\begin{array}{l} \cancel{1} \mp 2z^{-1} \pm z^{-2} \\ \hline 4z^{-1} - z^{-2} \\ 4z^{-1} \mp 8z^{-2} \pm 4z^{-3} \\ \hline 7z^{-2} - 4z^{-3} \\ 7z^{-2} - 14z^{-3} + 7z^{-3}. \\ \hline 10z^{-3} - 7z^{-3} \end{array}$$

$\therefore\ x(0)=1 \quad x(2)=7 \quad \cdots$

$\quad\ x(1)=4 \quad x(3)=10 \quad \cdots \Big\}$ ———— ①

② $\dfrac{X(z)}{z} = \dfrac{z+2}{(z-1)^2}$ (Partial fraction Method)

$$\frac{X(z)}{z} = \frac{A}{z-1} + \frac{B}{(z-1)^2}$$

$$B = (z+2)\Big|_{z=1} = 3.$$

$$A = \frac{d}{dz}(z+2)\Big|_{z=1} = 1.$$

$\therefore\ X(z) = \dfrac{z}{z-1} + \dfrac{3z}{(z-1)^2} =$ ~~u(k) + 3 k u(k)~~ *

for $k \geq 0$ $\quad x(k) = u(k) + 3k\,u(k)$

$\quad = 1 + 3k.$

$x(0) = 1 \quad x(2) = 7 \quad \cdots$

$x(1) = 4 \quad x(3) = 10 \quad \cdots \Big\}$ ———— ②

③ Cauchy's Method (or) Inversion Integral Method :-

$$X(z)\,z^{k-1} = \frac{(z+2)\,z^{k}}{(z-1)^2} \quad \forall \ k \text{ values.}$$

for all values of k $X(z)\,z^{k-1}$ has two poles at $z=1$.

$$\therefore \quad x(k) = K_1$$

$$K_1 = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d}{dz}\left[\cancel{(z-1)^2}\,\frac{(z+2)\,z^{k}}{\cancel{(z-1)^2}}\right]$$

$$= \lim_{z\to 1} \left(z^{k} + (z+2)\,k\,z^{k-1} \right)$$

$$K_1 = 1^{k} + 3k(1)^{k-1}$$

$$\therefore \quad x(k) = (1)^{k} + 3k(1)^{k-1}$$

for $k \geq 0$ $\quad x(k) = 1+3k$

$$\left.\begin{array}{ll} x(0) = 1 & x(2) = 7 \ \ldots \\ x(1) = 4 & x(3) = 10 \ \ldots \end{array}\right\} \quad - ③$$

Observation:-

from ①, ②, ③ . it is observed ~~whether~~ whichever may be the method followed, the final result is same in obtaining of inverse z-transform.

2.22

Modelling Discrete - Time Systems by Pulse Transfer function :-

(35) (26)

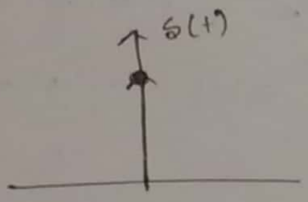
~~Impulse~~ - ~~Sampling~~ ~~and~~ ~~Data hold~~ :-

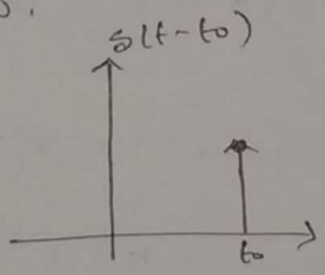
→ In Discrete - time Control Systems, while analyzing some signals are Continuous - time functions and other are Discrete. So there is a hybrid signal processing occurs in such a system.

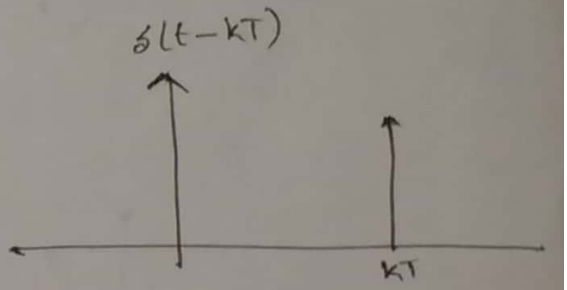
→ The analysis of systems which processes Discrete - time signal is carried by Z-transform. The Z-transform method is particularly useful for analysing and designing SISO, LTI, Discrete time control systems.

⇒ ① Impulse Sampling ~~and~~ ~~Data~~ ~~hold~~ :-

→ Mathematically an impulse is a signal having infinite amplitude with zero width, graphically shown by an arrow.

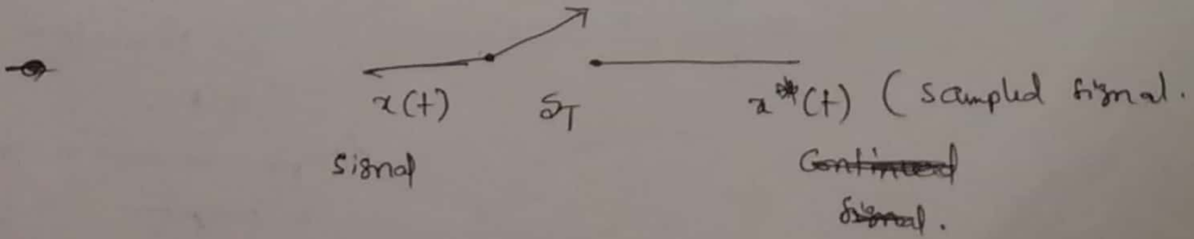


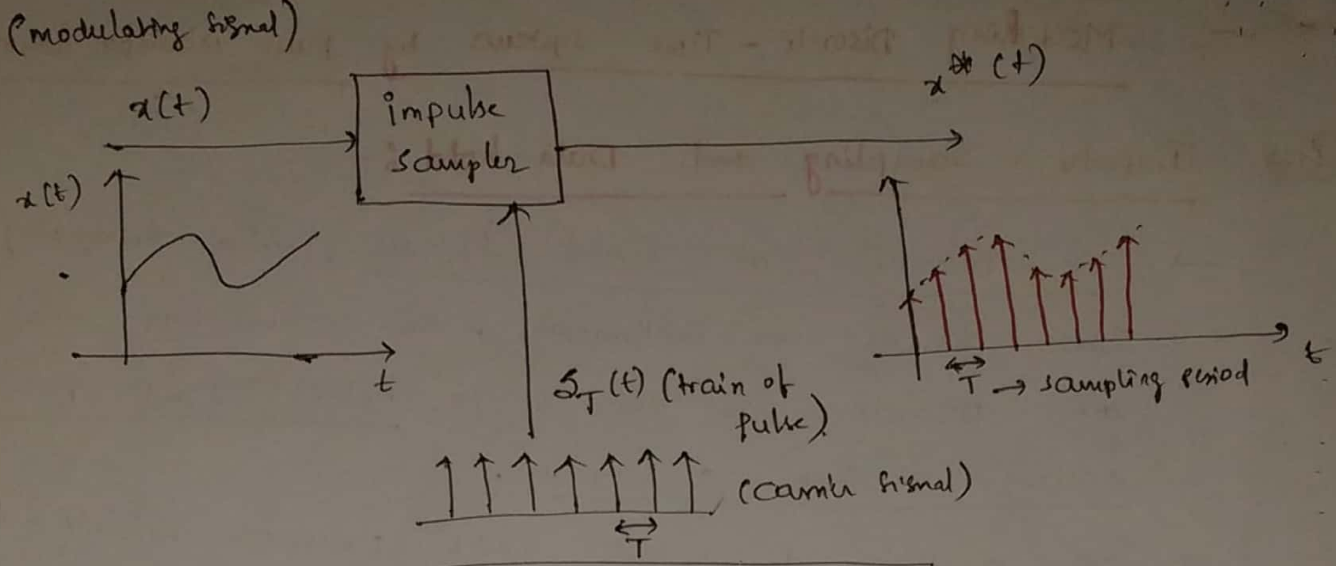




→ The impulse - sampled o/p of a signal x(t) is a sequence of impulses, with the strength of each impulse equal to the magnitude of x(t) at the corresponding instant of time

ie at $t = kT$ impulse $= x(kT)\,\delta(t - kT)$





$$\therefore \boxed{x^*(t) = \sum_{k=0}^{\infty} x(kT)\,\delta(t-kT)} \quad \text{i.e} \longleftarrow ①$$

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t-T) + x(2T)\delta(t-2T) + \dots + x(kT)\delta(t-kT) + \dots$$

→ Let's define impulse train as $\boxed{\delta_T(t) = \sum_{k=0}^{\infty} \delta(t-kT)}$

apply laplace to ①

$$X^*(s) = L[x^*(t)] = x(0)L[\delta(t)] + x(T)L[\delta(t-T)] + \dots + x(kT)L[\delta(t-kT)] + \dots$$

$$= x(0)\cdot(1) + x(T)e^{-Ts}(1) + \dots + x(kT)e^{-kTs} + \dots$$

$$\boxed{X^*(s) = \sum_{k=0}^{\infty} x(kT)\,e^{-kTs}} \longleftarrow ②$$

now if $\boxed{z = e^{Ts} \text{ then. } s = \frac{1}{T}\ln z.}$

$$② \Rightarrow \boxed{X^*(s)\Big|_{s=\frac{1}{T}\ln(z)} = \underbrace{\sum_{k=0}^{\infty} x(kT)\,z^{-k}.}_{\text{z-transform of } x(t).} = X(z)} \quad ③$$

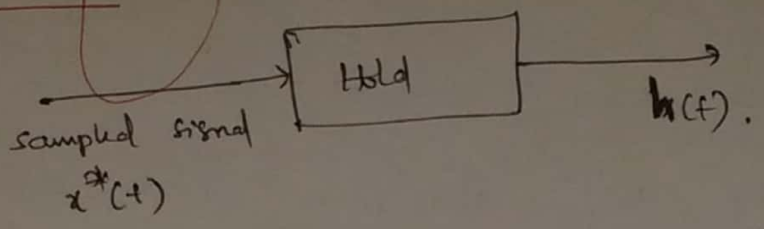
(refer motivation of using z-transform in ch-2 Pg 201.)

Observation :- from eqn ③ it is evident the Laplace transform of Impulse-sampled signal $x^*(t)$ is same to z-transform of signal $x(t)$

7.23

(36) (27)

⇒ Data-Hold circuit :-



→ A Hold circuit converts the sampled data signal into a Continuous-time signal, which approximately reproduces the Signal applied to the sampler.

ie $h(t) \simeq x(t)$.

→ The signal $h(t)$ during $kT \le t \le (k+1)T$ may be approximated by the following polynomial.

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + a_0 .$$

where $0 \le \tau < T$

if $n=1 \Rightarrow$ first-order Hold

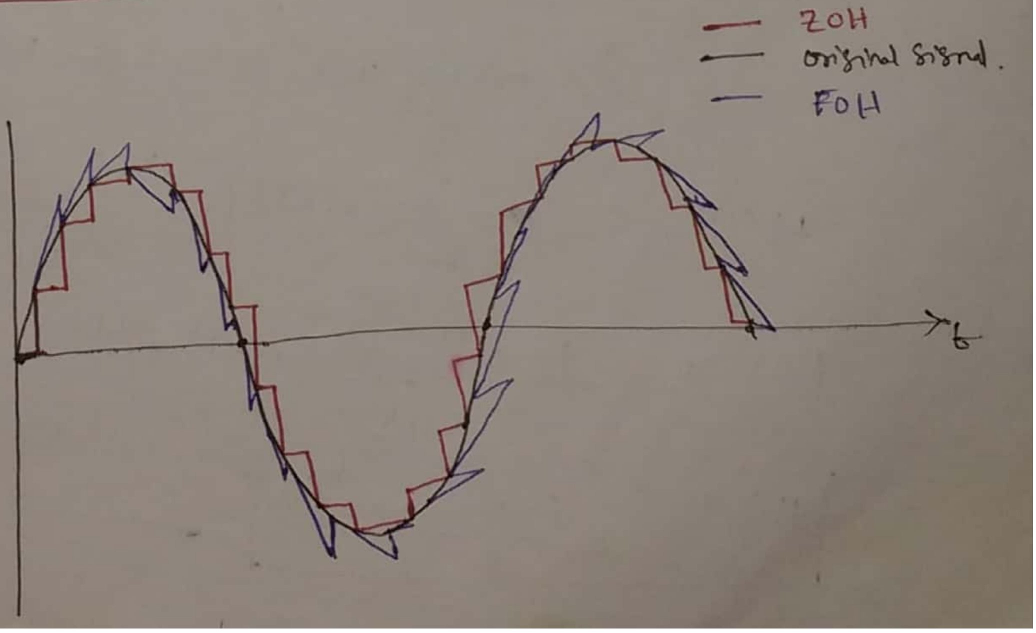
$n=0 \Rightarrow$ Zero-order Hold.

→ An nth order hold uses $(n+1)$ discrete data to generate original signal $h(kT+\tau)$. Higher the order better the

*→ Higher order hold gives good efficiency of signal reconstruction at the cost of time delay. This may be lead to stability problems.

⇒ ~~Zero-order Hold :-~~

Example



$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + --- + a_1\tau + \textcircled{a_0}$$

h(kT) must equal to x(kT) so.

$$\boxed{h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + --- + a_1\tau + x(kT)} \text{ —— } ①$$

→ Transfer function of Z.O.H :-

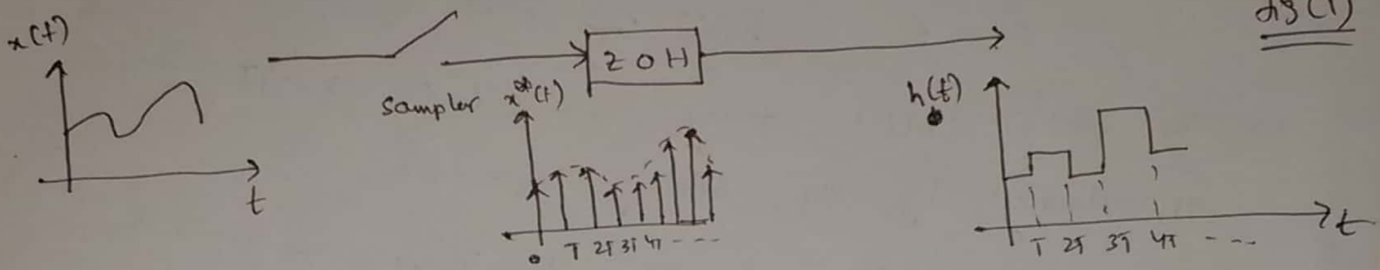


fig (1)

→ for a ZOH ~~$h(kT) = x(kT$~~

~~$h(kT+t) = x(kT)$~~ ~~$0 \le t \le T$~~.

$h(kT+\tau) = x(kT)$ $kT \le \tau \le (k+1)\tau$.

→ mathematical model of sampler + ZOH

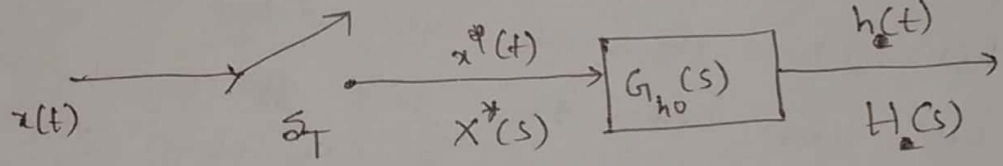


fig 2

→ ~~h(t)~~ = ~~x(t)~~ from fig (1)

$$h(t) = x(0)[u(t) - u(t-T)] + x(T)[u(t-T) - u(t-2T)] + ---$$

$$= \sum_{k=0}^{\infty} x(kT)[u(t-kT) - u(t-(k+1)T)]$$

$$H(s) = \sum_{k=0}^{\infty} x(kT)\left[\frac{e^{-kTs} - e^{-(k+1)Ts}}{s}\right]$$

$$H(s) = \frac{1-e^{-Ts}}{s} \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\boxed{H(s) = \frac{1-e^{-Ts}}{s} X^*(s)} \text{ ——— } ②$$

from fig ② and eqn ②

㊲

㉘

$$\boxed{G_{ho}(s) = \frac{1-e^{-Ts}}{s}}$$

***

$$\therefore \quad \boxed{H(s) = G_{ho}(s)\, X^*(s)}$$

** ie a real sampler and Z.O.H can be replaced by a mathematically equivalent continuous - time system that consists of an impulse sampler and a transfer function $G_{ho}(s) = \frac{1-e^{-Ts}}{s}$

Note T.F of F.O.H $G_{h1}(s) = \left(\frac{1-e^{-Ts}}{s}\right)^2 \left(\frac{1+Ts}{T}\right)$. ✓

2.24 ⓑ

(38)

(29)

⇒ calculating $X(z)$ from $X(s)$:-

→ assume $X(s)$ has ~~left~~ poles on left-half plane of s

and $X(s) = \frac{q(s)}{P(s)}$. Also assume $p(s)$ is of higher degree in s than $q(s)$. then

$$X(z) = \sum \left[\text{residue of } \frac{X(s)z}{z-e^{Ts}} \text{ at pole of } X(s) \right]$$

→ if $s_1, s_2 \dots s_m$ are the poles of $X(s)$, and if a pole at $s = s_i$ is a simple pole. then the Corresponding residue k_i is

$$\boxed{k_i = \lim_{s \to s_i} \left[(s - s_i) \frac{X(s)z}{z-e^{Ts}} \right]}$$

→ if a pole at $s = s_j$ is a multiple pole of order n_j then the residue at k_j is

$$\boxed{k_j = \frac{1}{(n_j - 1)!} \lim_{s \to s_j} \left[\frac{d^{n_j - 1}}{ds^{n_j - 1}} \left[(s - s_j)^{n_j} \frac{X(s)z}{z - e^{Ts}} \right] \right]}$$

**

Prev JNTU **Problem**

find $X(z)$ of $X(s) = \frac{4}{s^2(s+2)}$

Sol

$$X(z) = \sum \left[\text{residue of } \frac{X(s)z}{z - e^{Ts}} \text{ at pole. of } X(s) \right]$$

$X(s)$ has two poles at $s = 0$, one pole at $s = -2$

$$\therefore \text{ Residue at } s = 0 = k_1 = \frac{1}{(2-1)!} \lim_{s \to 0} \frac{d}{ds} \left[\cancel{s^2} \frac{4}{\cancel{s^2}(s+2)} \frac{z}{z - e^{Ts}} \right]$$

$$k_1 = \lim_{s \to 0} \frac{(s+2)(z-e^{Ts})(0) - 4z\left[(s+2)(-Te^{-Ts}) + (z - e^{Ts})\right]}{(s+2)^2(z-e^{Ts})}$$

$$= \frac{\cancel{(2)}\,\cancel{(z-1)}\; - 4z\left[2(-T) + (z-1)\right]}{4(z-1)}$$

~~chapter 13~~

$$k_1 = \frac{-z(z-1-2T)}{(z-1)}$$

→ Residue at $s = -2 = k_2 = \lim_{s \to -2}\left[(s+\cancel{2})\frac{4}{s^2(s+\cancel{2})}\frac{z}{z-e^{Ts}}\right]$

$$k_2 = \frac{z}{z-e^{-2T}}$$

$\therefore \quad X(z) = k_1 + k_2$

$$\boxed{X(z) = \frac{(z+(2T+1))}{z-1} + \frac{z}{z-e^{-2T}}}$$

simplify further

⇒ obtaining z-transform of functions involving the Term $\frac{1-e^{-Ts}}{s}$:—

let $X(s) = \frac{1-e^{-Ts}}{s}G(s)$ then

$$X(z) = z[X(s)] = (1-z^{-1})\,z\left(\frac{G(s)}{s}\right)$$

Problem

find $X(s) = \frac{1-e^{-Ts}}{s}\frac{1}{s+1}$.

$$X(z) = z\left[\frac{1-e^{-Ts}}{s}\frac{1}{s+1}\right]$$

$$= (1-z^{-1})\,z\left[\frac{1}{s(s+1)}\right]$$

$z\left[\frac{1}{s(s+1)}\right]$ = Residue at $s=0$ + Residue at $s=-1$.

$$\lim_{s\to 0}\frac{1}{s+1}\frac{z}{z-e^{Ts}} + \lim_{s\to -1}\frac{1}{s}\frac{z}{z-e^{Ts}}$$

$$= \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$

$$\therefore \quad X(z) = (1-z^{-1})\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right]$$

225 ①

-: Sampling Theorem :-

(Signal Re-Construction).

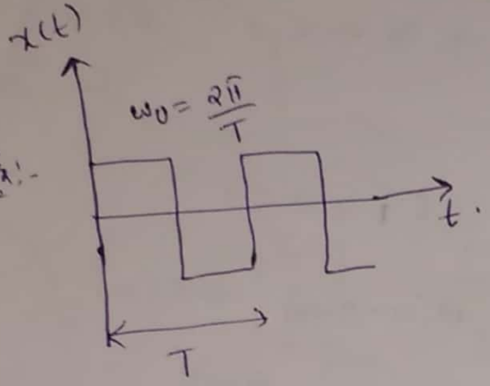
⇒ **Background :-**

→ **Exponential Fourier Series of Periodic Signals :-**

Let $x(t)$ be a periodic signal.

then $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$.

Ex:-

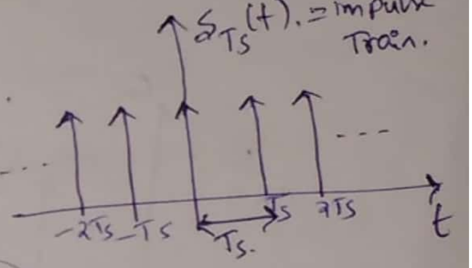


where C_n = Fourier series Co-efficient.

$$C_n = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-j\omega t}\, dt.$$

w_0 = frequency in rad/sec of $x(t)$.

→ **Exponential Fourier series of $\delta_{T_s}(t)$.**



$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}.$$

Fourier Series Co-efficient of $\delta_{T_s}(t)$ $= C_n = \frac{1}{T_s}\underbrace{\int_{\langle T_s\rangle} \delta_{T_s}(t)\, e^{jm\omega t}}_{\to 1} = \frac{1}{T_s}.$

→ **Fourier transform :-**

Fourier series is applicable to periodic signal

Fourier transform is applicable to aperiodic signal.

$$F.T[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

→ **Fourier transform of periodic Signals :-**

The Fourier transform of band limited periodic signal $x(t)$ is equal to $2\pi C_n$. times shifted impulses. $\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$.

when C_n = Fourier series co-efficient of $x(t)$.

Ex 2 find F.T of $\cos \omega_0 t$?

Sol first find Fourier series of $\cos\omega_0 t$.

$$x(t) = \cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}.$$

$$x(t) = 0.5\, e^{j\omega_0 t} + 0.5\, e^{-j\omega_0 t} \quad — ①$$

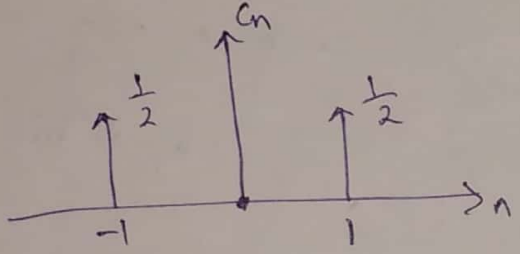
but $$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}.$$

$$= \dots + c_{-1} e^{-j\omega_0 t} + c_0 + c_1 e^{j\omega_0 t} + \dots \quad ②$$

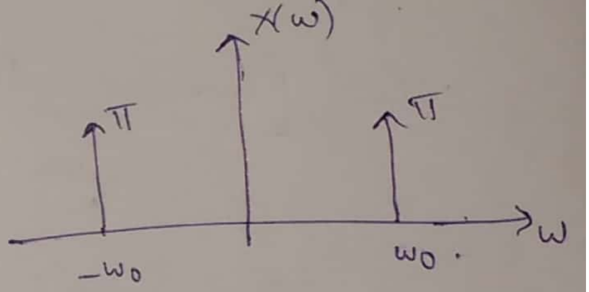
Comparing ①, ②

$$c_0 = 0, \quad c_{-1} = 0.5, \quad c_1 = 0.5.$$

$$\therefore \text{F.T}[\cos\omega_0 t] = \sum_{n=-\infty}^{\infty} 2\pi c_n \delta(\omega - n\omega_0) = X(\omega)$$



X([illegible]) =



$$X(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + n\omega_0)]$$

→ Fourier transform of $\delta_{T_s}(t)$ i.e impulse train :-

$$\text{F.T}[\delta_{T_s}(t)] = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0).$$

$$c_n = \frac{1}{T_s} \text{ for } \delta_{T_s}(t).$$

$$\therefore \text{F.T}[\delta_{T_s}(t)] = \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_0 n) = \delta_{\omega_s}(\omega).$$

→ Convolution property of F.T :-

let $x_1(t) \xleftrightarrow{F.T} X_1(\omega)$.

$x_2(t) \longleftrightarrow X_2(\omega)$

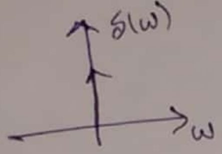
$$\underbrace{x_1(t)\, x_2(t)}_{\text{multiplication}} \longleftrightarrow \frac{1}{2\pi}\left[\underbrace{X_1(\omega) * X_2(\omega)}_{\text{convolution}}\right]$$

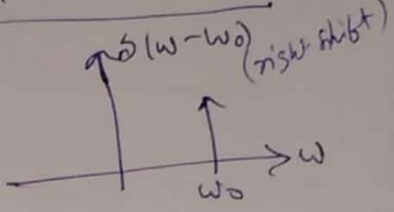
Convolution with impulse signal :-

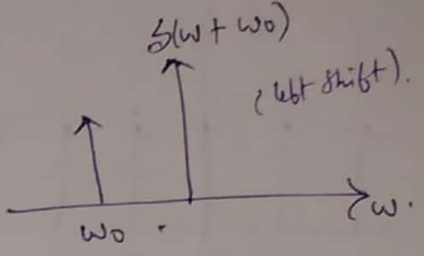
The Convolution ~~with~~ of a signal with impulse will be the signal. The convolution of a signal with the shifted impulse will be the same signal but shifted.

Note

Shifting operation on signals :-







⇒

Sampling Theorem :-

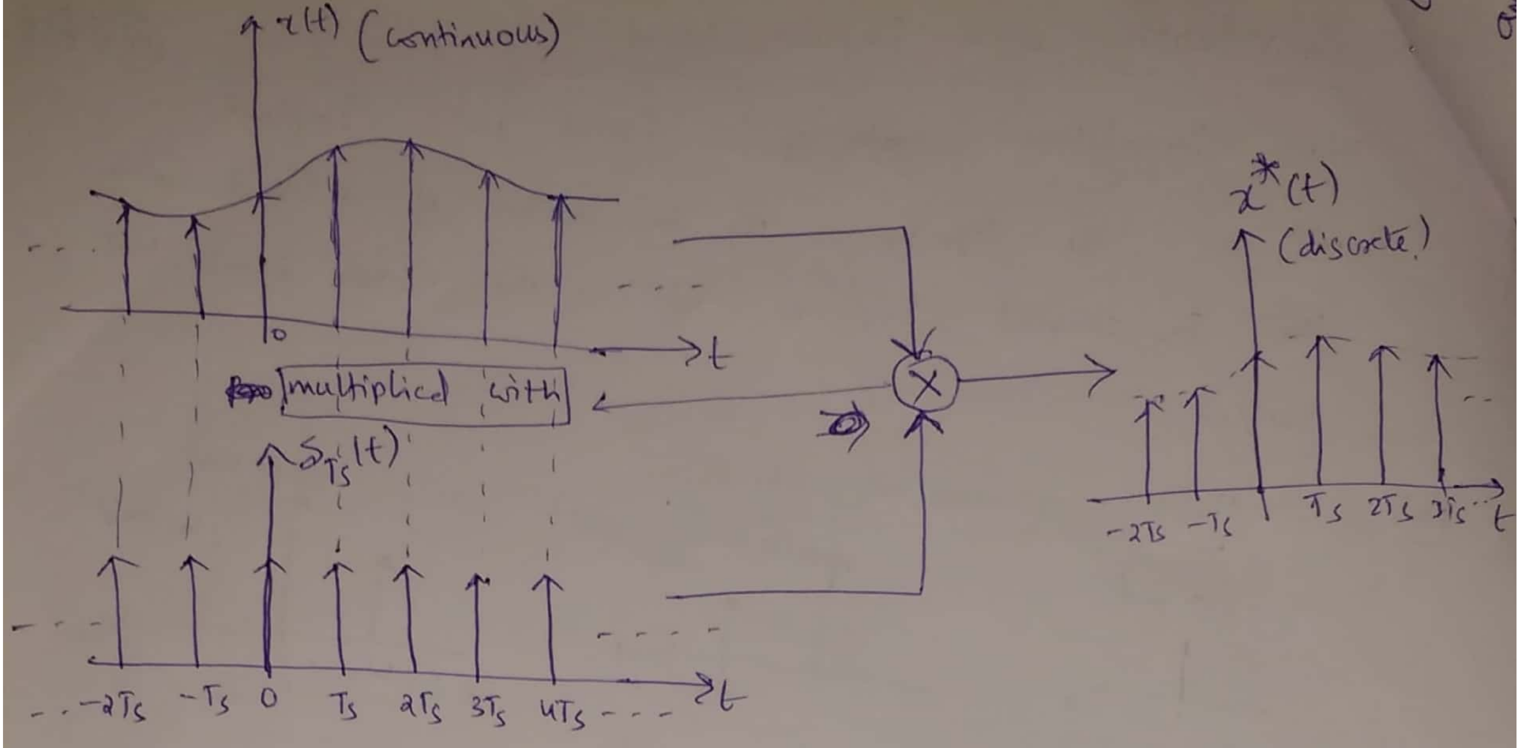
"The Reconstruction of sampled signal (band limited) is possible if the sampling frequency is atleast double to the maximum frequency component of the original signal."

The rate at which the signal is sampled to avoid aliasing is called Nyquist rate i.e $W_s \geq 2W_1$.

where W_s = sampling frequency
W_1 = maximum frequency component.

Proof

Sampling is the process of converting a ~~Continuous~~ Continuous time signal into discrete signal. It is done by multiplying a continuous signal $x(t)$ with train of impulses. $\delta_{T_s}(t)$



Let X(ω) is the Fourier Transform of x(t)

$$X(\omega) = F\cdot T\{x(t)\}.$$

$\delta_{\omega_s}(\omega)$ is the Fourier Transform of $\delta_{T_s}(t)$ ie impulse train

$$\delta_{\omega_s}(\omega) = F\cdot T[\delta_{T_s}(t)].$$

$$\boxed{\delta_{\omega_s}(\omega) = \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0).}$$

Sampled signal $x^*(t) = x(t)\,\delta_{T_s}(t)$.

$$F\cdot T[x^*(t)] = F\cdot T[x(t)\,\delta_{T_s}(t)].$$

by convolution property

$$F\cdot T[x^*(t)] = \frac{1}{2\pi}[X(\omega) * \delta_{\omega_s}(\omega)].$$

$$X^*(\omega) = \cancel{\frac{1}{2\pi}}\left[X(\omega) * \frac{\cancel{2\pi}}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s)\right].$$

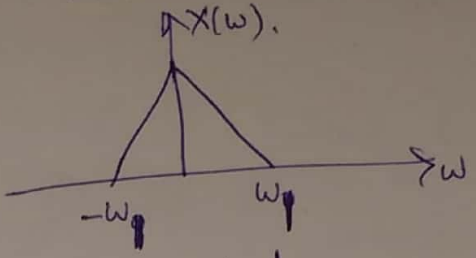
$$X^*(\omega) = X(\omega) * \sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s).$$

2.26 B

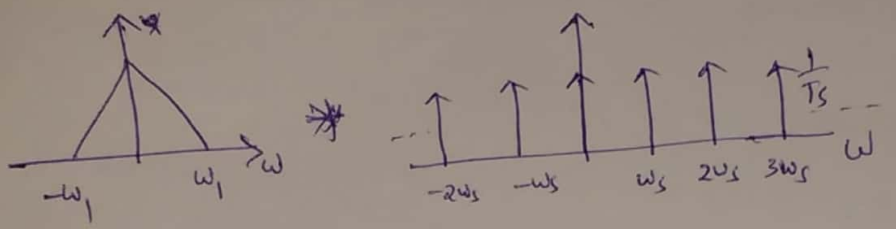
let's assume x(t) is band limited signal

and let's take X(ω) as

$X(\omega) =$



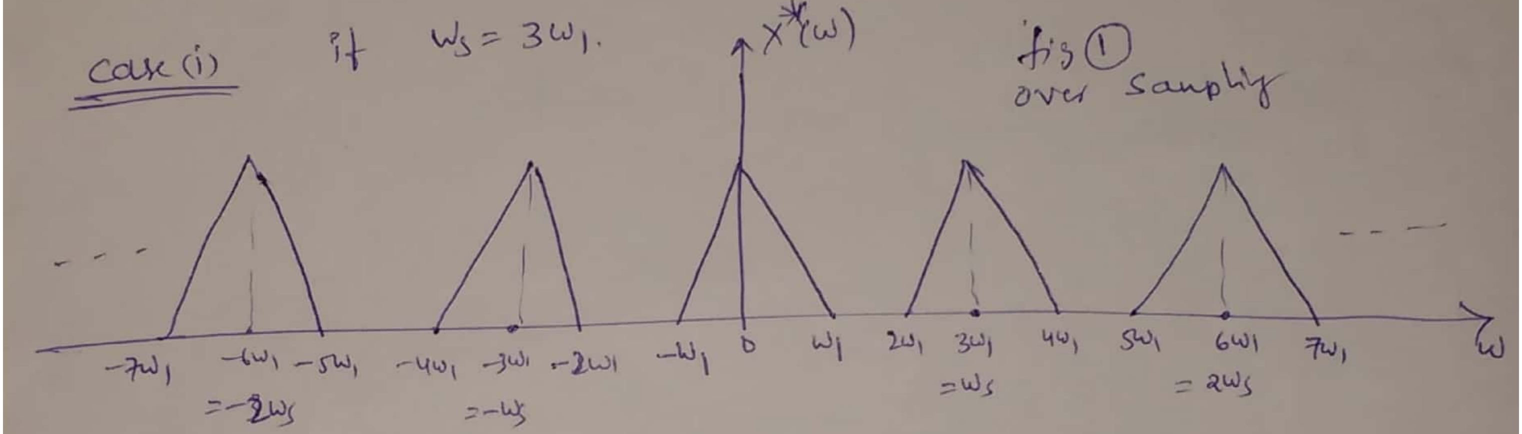
$\therefore X(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) =$ [a signal Convolution with impulses will be the same signal but shifted]



$= X^*(\omega)$

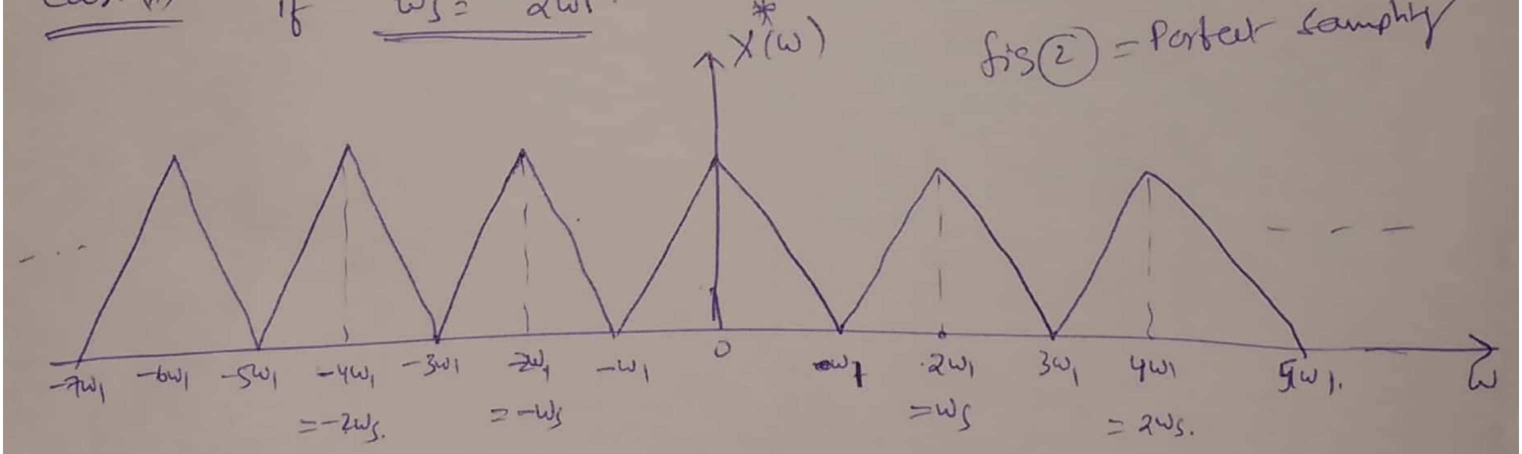
case (i) if $\omega_s = 3\omega_1$.

fig ① over sampling



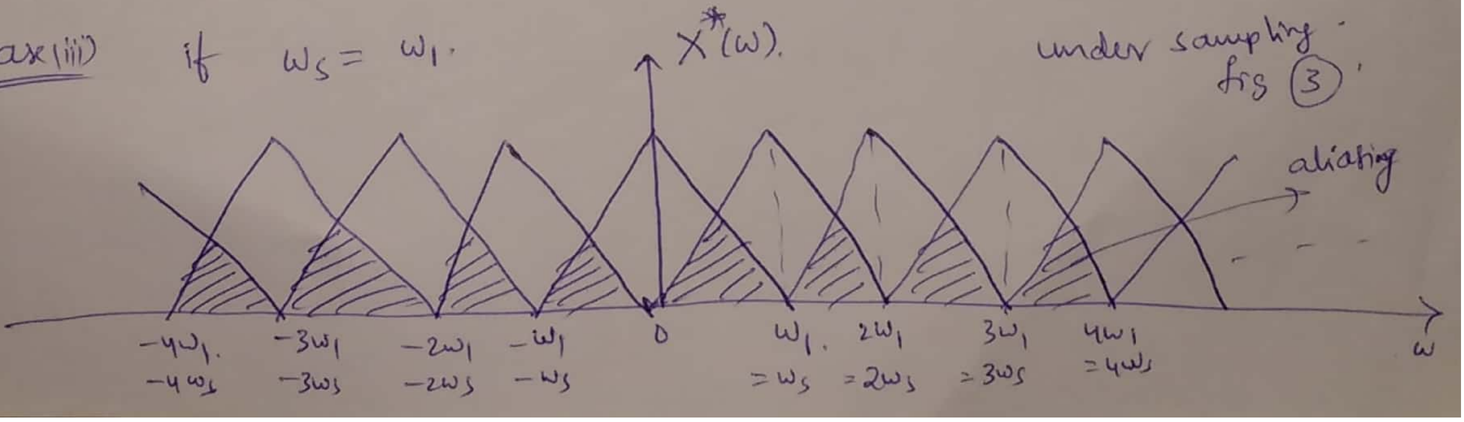
case (ii) if $\omega_s = 2\omega_1$.

fig ② = Perfect sampling



case (iii) if $\omega_s = \omega_1$.

under sampling. fig ③



→ if $w_s \geq 2w_1$ (fig ① & ②) there is no overlap of frequency bands of $X(w)$, so the reconstruction of signal is possible.

→ but if $w_s < 2w_1$ (fig ③) we see there occurs overlap of frequency bands of $X(w)$. i.e. The corresponding frequency components in time domain will not be retrieved (or reconstructed) as overlap in frequency domain occurs.

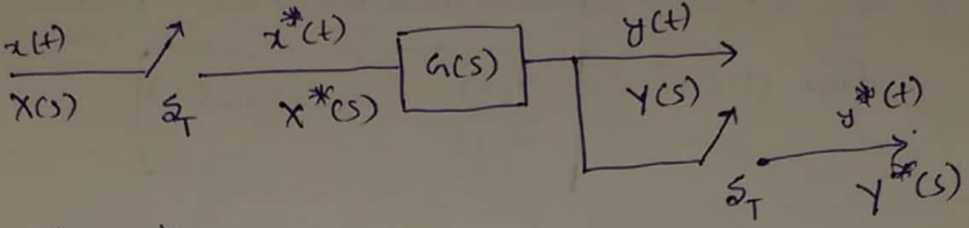
So graphically we proved that minimum rate at which sampling must be done in order to avoid aliasing effect so that the original signal can be reconstructed from sampled signal.

⇒ General procedure to find P.T.F :-

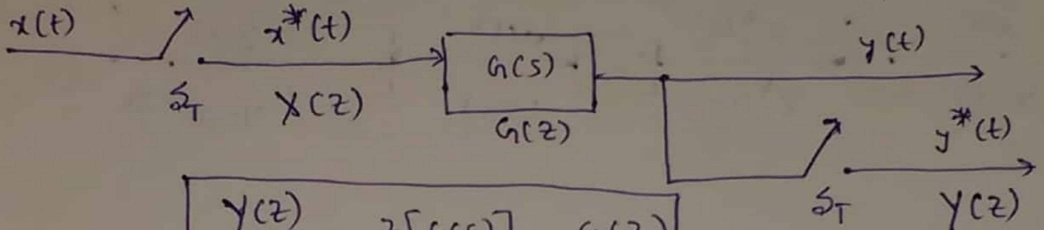
㊶ ㉜

① → if
$$\boxed{\begin{array}{l} Y(s) = G(s)\,X^*(s) \quad \text{then} \\ Y^*(s) = G^*(s)\,X^*(s) \end{array}}$$

$*$ → means signals are impulse sampled.



② →



$$\boxed{\frac{Y(z)}{X(z)} = Z[G(s)] = G(z)}$$; $$\boxed{\cancel{G(s) = \frac{Y(s)}{X(s)}}}$$

~~***~~

③ → if
$$\boxed{\begin{array}{l} Y(s) = G(s)\,X^*(s) \\ Y^*(s) = G(s)^*\,X^*(s) \end{array}}$$ $\longrightarrow \;\Rightarrow\; Y(z) = G(z)\,X(z)$

if
$$\boxed{\begin{array}{l} Y(s) = G(s)\,X(s) \\ Y^*(s) = [GX(s)]^* \end{array}}$$ $\longrightarrow Y(z) = GX(z)$

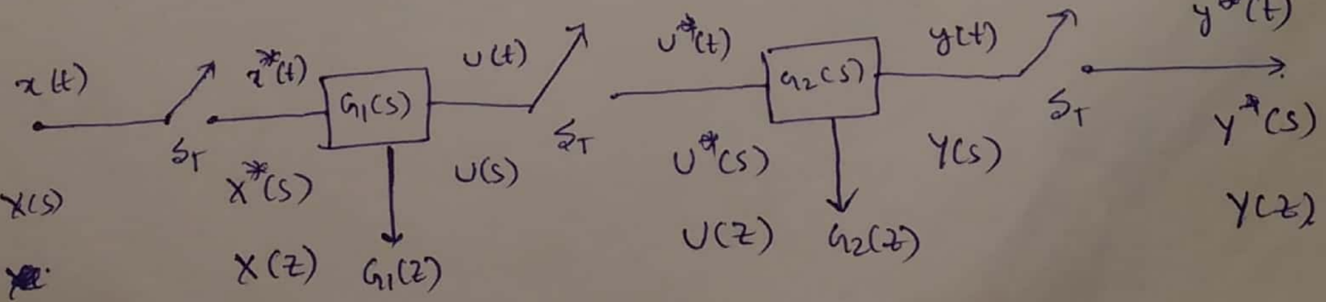
& $G(z)\,X(z) \neq GX(z)$

in term of z-transforms

~~Bo~~ ~~Find PTF~~

④ → P.T.F of cascaded Elements :-

ⓐ if samplers are synchronised and have the same sampling Period



~~$U(s) = G(s) X^*(s) \Rightarrow U^*(s) = G_1^*(s) X^*(s) \Rightarrow U(z) = G$~~

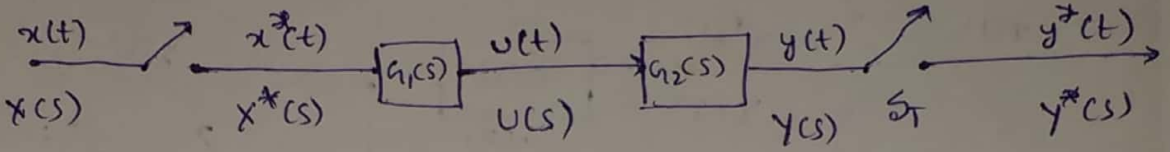
~~$Y(s) = H(s) X^*(s) \Rightarrow Y^*(s) = H^*(s) U^*(s)$~~

$U(s) = G_1(s) X^*(s) \Rightarrow U^*(s) = G_1^*(s) X^*(s) \Rightarrow U(z) = G_1(z) X(z)$ — ①

$Y(s) = G_2(s) U^*(s) \Rightarrow Y^*(s) = G_2^*(s) U^*(s) \Rightarrow Y(z) = G_2(z) U(z)$ — ②

from ① & ② $\boxed{\frac{Y(z)}{X(z)} = G_1(z) G_2(z)}$ — ①

ⓑ



~~$Y(s) = X^*(s) G_1(s) G_2(s))$~~

$U(s) = X^*(s) G_1(s)$

$Y(s) = G_2(s) U(s).$

$Y(s) = G_2(s) G_1(s) X^*(s)$

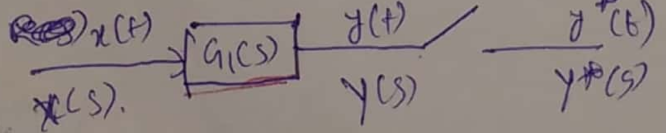
$Y^*(s) = [G_1 G_2(s)]^* X^*(s)$

$Y(z) = G_1 G_2(z) X(z)$

$$\boxed{P.T.F = \frac{Y(z)}{X(z)} = G_1 G_2(z)}$$ — ②

Observation from the two different arrangements it is observed that the presence of sampler makes different P.T.F.s. since $G_1G_2(z) \neq G_1(z) \cdot G_2(z)$

ⓒ



$Y(s) = X(s) G(s)$

$Y^*(s) = X G^*(s)$

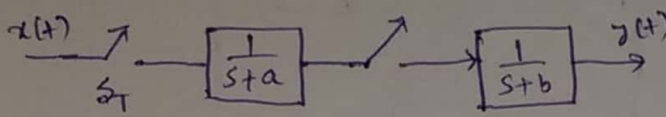
$Y(z) = X G(z)$

Q.28

Q20 understand the difference between $G_1G_2(z)$ and $G_1(z)\,G_2(z)$.

(42)

(33)

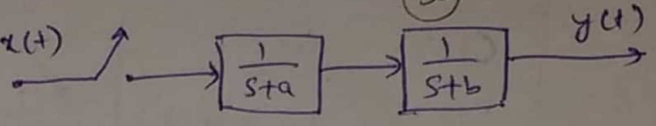


$$P.T.F = G_1(z)\,G_2(z)$$

$$= z\left[\frac{1}{s+a}\right]\cdot z\left[\frac{1}{s+b}\right]$$

$$= \left[\text{Residue of}\left(\frac{1}{s+a}\,\frac{z}{z-e^{Ts}}\right)\right]\left[\text{Res}\,\frac{1}{(s+b)}\,\frac{z}{z-e^{Ts}}\right]$$

$$\boxed{G_1(z) = \frac{1}{1-e^{-aT}z^{-1}}\cdot\frac{1}{1-e^{-bT}z^{-1}}.}$$



$$P.T.F = G'(z) = z\left[\frac{1}{s+a}\cdot\frac{1}{s+b}\right]$$

$$= z\left[\frac{1}{b-a}\left[\frac{1}{s+a}-\frac{1}{s+b}\right]\right]$$

$$G'(z) = \frac{1}{b-a}\left[\frac{1}{1-e^{-aT}z^{-1}}-\frac{1}{1-e^{-bT}z^{-1}}\right]$$

***

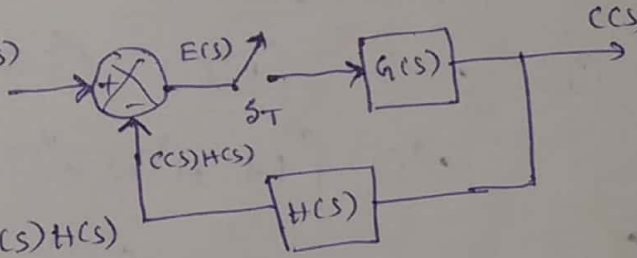
$$\boxed{\begin{aligned} \text{i.e}\quad G_1(z)\,G_2(z) &= z[G_1(s)]\cdot z[G_2(s)] \\ G_1G_2(z) &= z[G_1(s)\,G_2(s)] \end{aligned}}$$

⇒ P.T.F of -ve feedback closed loop system :-

ⓐ No out-put sampler :-

$$E(s) = R(s) - C(s)H(s) \quad \text{— ①}$$

$$C(s) = E^*(s)\,G(s) \quad \text{— ②}$$



①, ② ⇒ $E(s) = R(s) - E^*(s)\,G(s)H(s)$

$$E^*(s) = R^*(s) - [GH(s)]^*\,E^*(s)$$

$$E^*(s) = \frac{R^*(s)}{1+GH^*(s)} \Rightarrow E(z) = \frac{R(z)}{1+GH(z)} \quad \text{— ③}$$

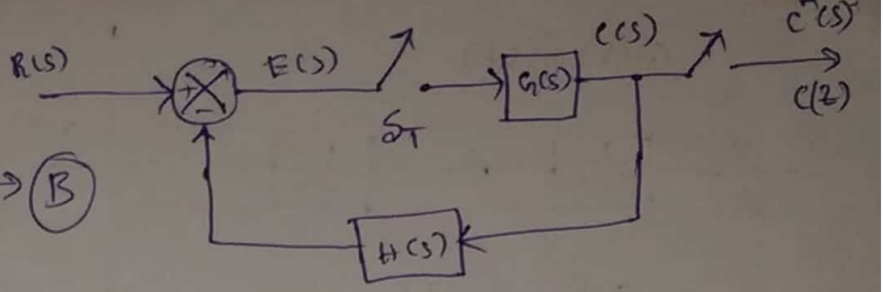
② ⇒ $C^*(s) = E^*(s)\,G^*(s).$

from ③ ⇒ $C^*(s) = \dfrac{R^*(s)\,G^*(s)}{1+GH^*(s)} \Rightarrow C(z) = \dfrac{R(z)\,G(z)}{1+GH(z)}$

Closed loop P.T.F = $\boxed{\dfrac{C(z)}{R(z)} = \dfrac{G(z)}{1+GH(z)}}$ — Ⓐ

(b) output sampler outside the loop :-

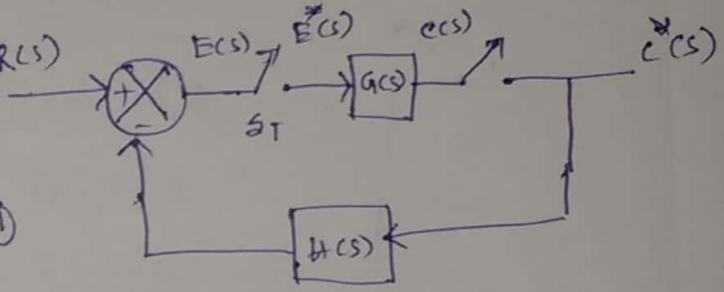
$$\boxed{\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}} \rightarrow (B)$$



observation, from eqns (A) & (B) it is observed if an output sampler makes no difference if it presents the outside of the loop.

Closed loop P.T.F without o/p sampler = CL P.T.F with o/p sampler outside the loop.

(C) output sampler inside the loop :-



$$E(s) = R(s) - c^*(s)H(s)$$

$$E^*(s) = R^*(s) - c^*(s)\,H^*(s) \quad \text{—} ①$$

$$C(s) = E^*(s)\,G(s)$$

$$c^*(s) = E^*(s)\;G^*(s)$$

$$① \Rightarrow \; c^*(s) = [R^*(s) - c^*(s)H^*(s)]\,G^*(s)$$

$$c^*(s)\left[1 + H^*(s)\,G^*(s)\right] = G^*(s)\;R^*(s)$$

$$\frac{c^*(s)}{R^*(s)} = \frac{G^*(s)}{1 + G^*(s)\,H^*(s)}.$$

i.e C.L P.T.F $\boxed{\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)\,H(z)}} \rightarrow ©$

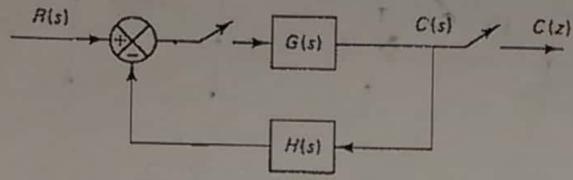
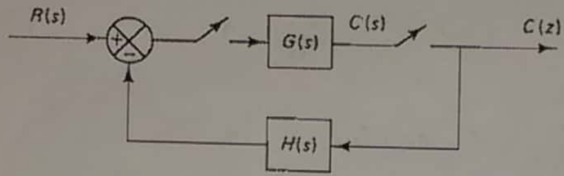
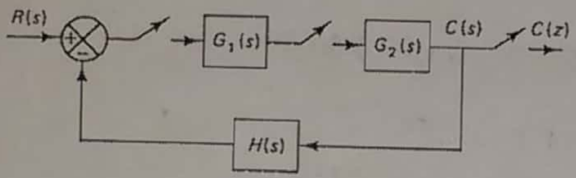
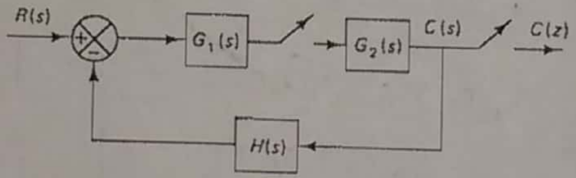
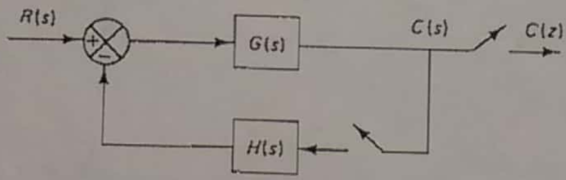
→ **observation** observe (A) & (B), (C).

(93)

(34)

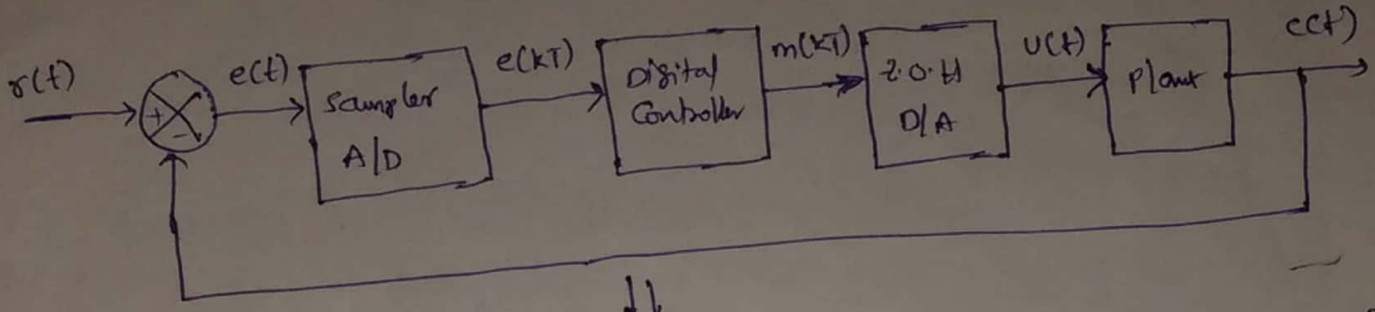
2:29

TABLE 3–1 FIVE TYPICAL CONFIGURATIONS FOR CLOSED-LOOP DISCRETE-TIME CONTROL SYSTEMS

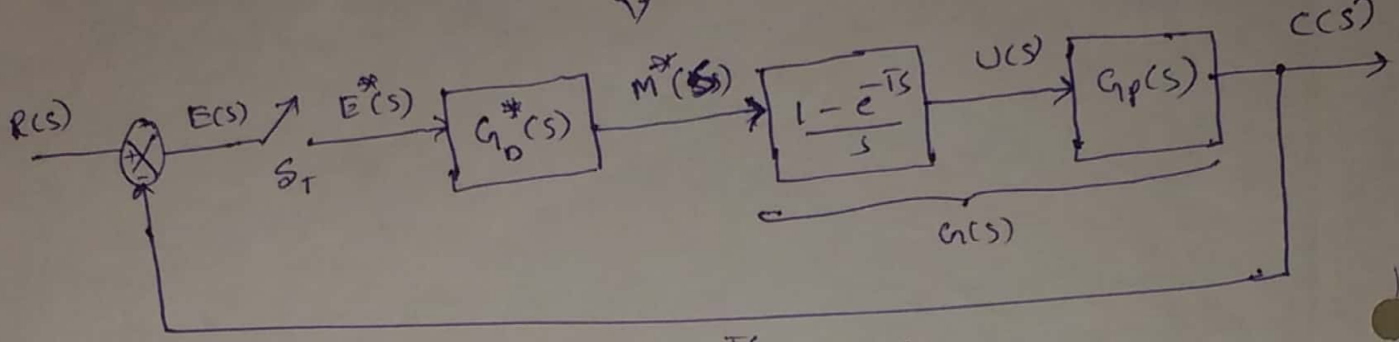
	$C(z) = \frac{G(z)R(z)}{1 + GH(z)}$	✓
	$C(z) = \frac{G(z)R(z)}{1 + G(z)H(z)}$	✓
	$C(z) = \frac{G_1(z)G_2(z)R(z)}{1 + G_1(z)G_2H(z)}$	✓
	$C(z) = \frac{G_2(z)G_1R(z)}{1 + G_1G_2H(z)}$ $G_1(z)$ [illegible]	✓
	$C(z) = \frac{GR(z)}{1 + GH(z)}$	✓

***

⇒ Closed loop P.T.F of a Digital Control System :-



⇓



Let $G(s) = \frac{1-e^{-Ts}}{s} G_p(s)$

$$C(S) = E^*(S) . G_D^*(S) G(S)$$

$$C^*(S) = E^*(S) G_D^*(S) G^*(S)$$

i.e $C(z) = E(z) G_D(z) G(z)$. —①

& $E(S) = R(S) - C(S)$

$$E^*(S) = R^*(S) - C^*(S).$$

$$E(z) = R(z) - C(z). \quad —②$$

①, ② ⇒ $C(z) = G_D(z) G(z) [R(z) - C(z)]$

$$\boxed{\frac{C(z)}{R(z)} = \frac{G_D(z) G(z)}{1 + G_D(z) G(z)}}$$

2, 7, 9, 25, 34, 55, 60, 62, 81, 86, 95, C4, C5, C6, C8, D3, D7

D8, E9, F5, F6, F9, G5, G9, H3

L - 2, 8, 10, 12, 14, 17, 26, 35

2.31

⇒ Pulse transfer function from difference equation :- (45) (35)

→ In continuous time control system a physical systems is modeled mathematically by Differential equations.

→ In Discrete time control system the systems are modeled as by "Difference eqns".

→ A LTI Discrete-time ~~Control~~ system is characterized by

$$x(k) + a_1 x(k-1) + \dots + a_n x(k-n) = b_0 u(k) + b_1 u(k-1) + \dots + b_n u(k-n)$$

$u(k) \rightarrow$ system i/p $x(k) \rightarrow$ system o/p.

$$\left.\begin{aligned}
x(k+4) &= z^4 X(z) - z^4 x(0) - z^3 x(1) - z^2 x(2) - z x(3) \\
x(k+3) &= z^3 X(z) - z^3 x(0) - z^2 x(1) - z x(2). \\
x(k+2) &= z^2 X(z) - z^2 x(0) - z x(1). \\
x(k+1) &= z X(z) - z x(0) \\
x(k-1) &= z^{-1} X(z) \\
x(k-2) &= z^{-2} X(z) \\
x(k-3) &= z^{-3} X(z).
\end{aligned}\right\} \text{mostly used while solving difference eqns.}$$

Problems

1. Solve difference eqn given by $x(k+2) + 3x(k+1) + 2x(k) = 0$; $x(0) = 0$, $x(1) = 1$.

Sol . apply z-transform

$$z^2 X(z) - z^2 x(0) - z x(1) + 3z X(z) - z x(0) + 2X(z) = 0.$$

$$X(z) = \frac{z}{z^2 + 3z + 2} = \frac{z}{(z+1)(z+2)}$$

$$X(z) = \frac{z}{z+1} - \frac{z}{z+2}$$

$$\boxed{x(k) = (-1)^k u(k) - (-2)^k u(k)}$$

Q $x(k+2) - 0.1\,x(k+1) - 0.2\,x(k) = r(k+1) + r(k).$

with $r(k) = u(k)$; $x(0) = 0$ and $x(1) = 0$

(i) find P.T.F. (ii) obtain the o/p $x(k)$ for an input of step.

Sol (i) $x(k+2) - 0.1\,x(k+1) - 0.2\,x(k) = r(k+1) + r(k).$

applying z-transform

$$z^2X(z) - z^2\cancelto{0}{x(0)} - z\cancelto{0}{x(1)} - 0.1\,zX(z) + z\cancelto{0}{x(0)} - 0.2\,X(z) = zR(z) - \cancelto{0}{z\,r(0)} + R(z)$$

$$X(z)\left[z^2 - 0.1z - 0.2\right] = R(z)\left[z+1\right]$$

$$\frac{X(z)}{R(z)} = \frac{z+1}{z^2 - 0.1z + 0.2} \longleftarrow \text{P.T.F.}$$

(ii) o/p to step i/p.

$$X(z) = \frac{(z+1)\,Z[u(k)]}{z^2 - 0.1z - 0.2} = \frac{10(z+1)}{10z^2 - z - 2}\;\frac{z}{z-1}$$

$$X(z) = \frac{10z(z+1)}{(z-0.5)(z+0.4)(z-1)}$$

$$\frac{X(z)}{z} = \frac{A}{z-0.5} + \frac{B}{z+0.4} + \frac{C}{z-1}$$

$$A = \left.\frac{10z(z+1)}{(z+0.4)(z-1)}\right|_{z=0.5} = \frac{10\times\cancel{1.5}\,3}{\underset{0.3}{\cancel{0.9}}(-0.5)} = \cancel{-16.6}\;\underline{\underline{-33.2}}$$

$$B = \left.\frac{10z(z+1)}{(z-0.5)(z-1)}\right|_{z=-0.4} = \frac{-10(0.6)}{(-0.9)(-1.4)} = \cancel{49}\;-4.7$$

$$C = \left.\frac{10z(z+1)}{(z-0.5)(z+0.4)}\right|_{z=1} = \frac{10\times 2}{0.5\times 1.4} = \frac{20}{0.7} = 28.57$$

$$X(z) = \frac{z(-33.2)}{z-0.5} + \frac{z(-4.7)}{z+0.4} + \frac{28.57\,z}{z-1}$$

$$= -33.2\,(0.5)^k u(k) \cancel{+} 4.7\,(-0.4)^k + 28.57\,\underline{\underline{u(k)}}$$

Ex find the z-transform of $f(k) = (0.1)^k u_s(k) + 0.5\,k\,(0.1)^{k-1} u_s(k-1)$. ③

(36) 2.32

$$Z.T\left[(0.1)^k u_s(k)\right] = \frac{z}{z-0.1}.$$

$$Z.T\left[(0.1)^{k-1} u_s(k-1)\right] = z^{-1}\frac{z}{z-0.1} = \frac{1}{z-0.1}.$$

$$Z.T\left[k\,(0.1)^{k-1} u_s(k-1)\right] = -z\frac{d}{dz}\frac{1}{(z-0.1)}$$

$$= -z.\frac{-1}{(z-0.1)^2}.$$

$$= \frac{z}{(z-0.1)^2}.$$

Ex find the Inverse z-transform of $F(z) = \frac{z+1}{(z-0.1)^2}$.

$$F(z) = \frac{z}{(z-0.1)^2} + \frac{1}{(z-0.1)^2}$$ (60)

$$F(z) = \frac{A}{(z-0.1)} + \frac{B}{(z-0.1)^2}$$

$$B = \left|\frac{z+1}{}\right|_{z=0.1} = \underline{\underline{1.1}}$$

$$z+1 = (z-0.1)^2 A + (z-0.1)B.$$

$$0.01A - 0.1B = 1 \Rightarrow. \quad A = \frac{1+0.1B}{0.01} = \frac{1.11}{0.01} = \underline{\underline{111}}$$

$$\therefore F(z) = z^{-1}\left[\frac{111\,z}{z-0.1}\right] + z^{-1}\left[\frac{1.1\,z}{(z-0.1)^2}\right]$$

we know $a^k \longleftrightarrow \frac{z}{z-a}$

$$k \longleftrightarrow \frac{z}{(z-1)^2}.$$

$$a^k k \longleftrightarrow \frac{z/a}{\left(\frac{z}{a}-1\right)^2} = \frac{1}{a}\frac{az}{(z-a)^2}.$$

$$\therefore f(k) = 111\,(0.1)^k - \frac{1.1.K(0.1)^k}{0.1}.$$

$$f(k) = 111(0.1)^k - 11k(0.1)^k.$$

② ~~plane angle.~~

~~solid angle.~~

① ~~Luminous flux.~~

② ~~Luminous Intensity~~

② ~~Lumens~~

② ~~Candle power~~

② ~~Illumination~~

③ ~~Bright~~

②.

Pr find Inverse z-Transform.

$$F(z) = \frac{2z}{z^2 - 1.2z + 0.5}$$

Sol

$$\frac{F(z)}{z} = \frac{2}{(z - 0.6 - j0.374)(z - 0.6 + j0.374)}$$

$$\frac{F(z)}{z} = \frac{K}{(z - 0.6 - j0.374)} + \frac{K^*}{(z - 0.6 + j0.374)}$$

$$F(z) = \frac{Kz}{(z - 0.6 - j0.374)} + \frac{K^* z}{(z - 0.6 + j0.374)}$$

$$K = \frac{2}{(z - 0.6 + j0.374)}\Bigg|_{z = 0.6 + j0.374}$$

$$K = \frac{2}{j0.748} =$$ [illegible]

$$K = 6.27 \angle -90^\circ$$

$$|K| = 6.27 \qquad \angle K = -90^\circ$$

$$\text{Pole } P = 0.6 - j0.374 = 0.707 \angle -31.93$$

$$|P| = 0.707 \qquad \angle P = \angle -31.93$$

$$\therefore \quad f(k) = A e^{-aTk} \cos(bTk + \theta)$$

where $A = 2|K_1| \qquad \theta = \angle K_1$

$A = 2 \times 6.27 \qquad \theta = -90^\circ$

$$-aT = \ln|P_1| \Rightarrow -aT = \ln 0.707$$

$$= -0.346$$

$$bT = \angle P_1 = -31.93^\circ \times \frac{\pi}{180} = -0.77\pi$$

$$\therefore \quad f(k) = 12.54\, e^{-0.346k} \cos(0.77\pi k + \pi/2)$$

②

㊲ 2.33

ⓑ $f(t) = \sin 2t$

$$f(kT) = \sin 2kT$$

$$= \frac{e^{j2kT} - e^{-j2kT}}{2j}$$

$$F(z) = \frac{1}{2j}\left[z \cdot T\left[e^{j2kT} \right] - z \cdot T\left[e^{-j2kT} \right] \right]$$

we know if . $x(k) = a^k$.

then $X(z) = \frac{z}{z-a}$

$$\therefore F(z) = \frac{1}{2j}\left[\frac{z}{z-e^{j2T}} - \frac{z}{z-e^{-j2T}} \right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{z^2} - ze^{-j2T} - \cancel{z^2} + ze^{j2T}}{z^2 - z(e^{j2T} + e^{-j2T}) + 1} \right]$$

$$F(z) = \frac{z \sin 2T}{z^2 - 2z \cos 2T + 1}.$$

ⓒ $f(t) = t \sin \omega t$

$$f(kT) = kT \sin \omega kT$$

we know $x(k) \longleftrightarrow X(z)$.

then $k^m x(k) \longleftrightarrow \left(-z \frac{d}{dz}\right)^m X(z)$.

$\therefore$ $x(kT) = \sin \omega kT$

$$X(z) = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$$

$$(kT)\, x(kT) = -z \frac{d}{dz}\left[\frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} \right]$$

$$= -z \left[\frac{(z^2 - 2z\cos\omega T + 1)(\sin \omega T) - z\sin\omega T(2z - 2\cos\omega T)}{(z^2 - 2z\cos\omega T + 1)^2} \right]$$

$$= -z\left[\frac{z^2 \sin\omega T - \cancel{2z\cos\omega T \sin\omega T} + \sin\omega T - 2z^2 \sin\omega T + \cancel{2z\cos\omega T \sin\omega T}}{(z^2 - 2z\cos\omega T + 1)^2} \right]$$

$$= \frac{-z(\sin\omega T - z^2 \sin\omega T)}{(z^2 - 2z\cos\omega T + 1)^2}$$

ⓓ $f(t) = e^{-2(t-T)} u_s(t-T)$

$f(kT) = e^{-2(kT-T)} u_s(kT-T)$.

we know $e^{-2kT} u_s(kT) \longleftrightarrow \frac{z}{z-e^{-2T}}$

The property $x(k) \longleftrightarrow X(z)$

$x(k-n) \longleftrightarrow z^{-n} X(z)$.

$\therefore \quad e^{-2(kT-T)} u_s(kT-T) \longleftrightarrow z^{-1} \cdot \frac{z}{z-e^{-2T}}$.

$\therefore \quad z \cdot T[f(k)] = z \cdot T\left[e^{-2(t-T)} u_s(t-T)\right] = \cancel{2} \frac{1}{z-e^{-2T}}$.

ⓔ $f(t) = e^{-at} \sin 2t$.

Sol = $x(t) = \sin 2t$

$X(z) = \frac{z \sin 2T}{z^2 - 2z \cdot \cos 2T + 1}$

~~a~~ $e^{-at} x(t) \longleftrightarrow X(e^{aT} z)$

$\therefore \quad z \cdot T\left[e^{-at} x(t)\right] = \frac{e^{aT} \sin 2T z}{e^{2aT} z^2 - 2e^{aT} z \cos \omega T + 1}$

Problems from unit-2.

Find z-transform

(38) 2.34

ⓐ $F(z) = \frac{z}{z^2+1} \Rightarrow \frac{F(z)}{z} = \frac{k_1}{z+j} + \frac{k_2}{z-j}$

$\Rightarrow F(z) = \frac{k_1 z}{z+j} + \frac{k_1^* z}{z-j}$

$k_1 = (z+j)\frac{1}{z^2+1}\Big|_{z=-j} = \frac{1}{-2j} = -0.5\angle -90^\circ$

$k_2 = (z-j)\frac{1}{(z-j)(z+j)}\Big|_{z=j} = \frac{1}{2j} = 0.5\angle -90^\circ$

$A = 2|k_1| = 2(-0.5) = -1.0$ $\qquad \theta = -90^\circ.$

Pole $= p = -j = 1\angle 90^\circ$ $\qquad \therefore -aT = \ln(1) \Rightarrow aT = 0.$

$bT = \angle -90^\circ \Rightarrow bT = -\pi/2.$

$\therefore f(k) = A e^{-aTk}\cos(bTk + \theta)$

$= -\cos(-\pi/2 k - \pi/2)$

$f(k) = \sin \frac{\pi}{2} k.$

ⓑ $F(z) = \frac{10z}{z^2-1}$ $\qquad A = \frac{10}{z+1}\Big|_{z=1} = 5$

$\frac{F(z)}{z} = \frac{10}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}.$ $\qquad B = \frac{10}{z-1}\Big|_{z=-1} = -5.$

$F(z) = \frac{5}{z+1} - \frac{5}{z-1}.$

$f(k) = 5[(-1)^k - (1)^k] = \begin{cases} 10 & k \text{ odd.} \\ 0 & k \text{ odd.} \end{cases}$

ⓒ $F(z) = \frac{A}{z} + \frac{B}{z-0.2}$ $\qquad F(z) = \frac{1}{z(z-0.2)}$

$A = \frac{1}{z-0.2}\Big|_{z=0} = -5.$

$B = \frac{1}{z}\Big|_{z=0.2} = 5.$

$\therefore F(z) = z^{-1}\left[\frac{5 \cdot z}{z-0.2}\right] - 5z^{-1}\left[\left(\frac{z}{z}\right)\right]$

$f(k) = 5\left[(0.2)^{k-1}u(k-1) - \delta(k-1)\right].$

Q. find I.Z-transform using power series.

$$F(z) = \frac{z(z+1)}{(z-1)(z^2-z+1)} = \frac{\cancel{z^2}\,(1+z^{-1})}{z\cancel{z^2}(1-z^{-1})(1-1/z+1/z^2)}$$

$$F(z) = \frac{z^{-1}+z^{-2}}{1-2z^{-1}+2z^{-2}-z^{-3}}$$

$$1-2z^{-1}+2z^{-2}-z^{-3}\,\big)\;\cancel{z^{-1}}+z^{-2}\;\big(\;z^{-1}+3z^{-2}+4z^{-3}-5z^{-5}+\cdots$$

$$\cancel{z^{-1}}\underset{+}{-}2z^{-2}\underset{-}{+}2z^{-3}\underset{+}{-}2z^{-4}$$

$$3\cancel{z^{-2}}-2z^{-3}-2z^{-4}$$

$$\cancel{3z^{-2}}\underset{+}{-}6z^{-3}\underset{-}{+}6z^{-4}\underset{+}{-}3z^{-5}$$

$$4\cancel{z^{-3}}-8\cancel{z^{-4}}+3z^{-5}$$

$$\cancel{4z^{-3}}\underset{+}{-}\cancel{8z^{-4}}\underset{-}{+}8z^{-5}\underset{+}{-}4z^{-6}$$

$$-5z^{-5}+4z^{-6}$$

$$\therefore f(k) = \{\underset{\uparrow}{0}, 1, 3, 4, -5, \ldots\}.$$

Q. find Z-transform of

ⓐ $f(t) = 3t\,e^{-at}$

$$f(kT) = 3kT\,e^{-akT}$$

we know $\quad x(kT) \longleftrightarrow X(z)$

$$(e^{-aT})^{k}\,x(kT) \longleftrightarrow X(e^{aT}z)$$

$\therefore$ ~~PE~~ Let $x(kT) = 3kT$

$$X(z) = \frac{3Tz}{(z-1)^2}$$

$$ZT\left[(e^{-aT})^{k}\,x(kT)\right] = \frac{3T\,e^{aT}z}{(e^{aT}z-1)^2}.$$

⇒ finding Inverse Z transform by Partial fractions when roots are complex

2.35

@

$$x(k) = A e^{akT} \cos(bkT + \theta) = \frac{A e^{akT}}{2}\left(e^{jbkT} e^{j\theta} + e^{-jbkT} e^{-j\theta}\right)$$

$$= \frac{A}{2}\left[e^{(aT + jbT)k} e^{j\theta} + e^{(aT - jbT)k} e^{-j\theta}\right]$$

$$= \frac{A}{2}\left[e^{j\theta} \frac{z}{z - e^{aT + jbT}} + \frac{e^{-j\theta} z}{z - e^{aT - jbT}}\right]$$

$$= \left(\frac{A e^{j\theta}}{2}\right) \frac{z}{z - e^{aT + jbT}} + \left(\frac{A e^{-j\theta}}{2}\right) \frac{z}{z - e^{aT - jbT}}$$

$$= \frac{k_1 z}{z - p_1} + \frac{k_1^* z}{z - p_1^*}$$

$$= k_1 \frac{z}{z - p_1} + \frac{k_1^* z}{z - p_1^*}$$

$p_1 = e^{aT} e^{jbT} = e^{aT} \angle bT \Rightarrow \boxed{\begin{aligned} aT &= \ln|p_1| \\ bT &= \arg p_1 \end{aligned}}$

& $k_1 = \frac{A e^{j\theta}}{2} = \frac{A}{2} \angle \theta \Rightarrow \boxed{\begin{aligned} A &= 2|k_1| \\ \theta &= \arg k_1 \end{aligned}}$

Pr 3 Solve the following difference eqn using z-transform method. ⑩

$$c(k+2) - 1.5c(k+1) + c(k) = 2u_s(k). \quad c(0)=0 \quad c(1)=1.$$

Sol

$$z^2C(z) - \cancel{z^2c(0)}^{\to 0} - zc(1) - 1.5zC(z) - \cancel{1.5zc(0)}^{\to 0} + C(z) = 2\frac{z}{z-1}.$$

$$C(z)\left[z^2 - 1.5z + 1\right] = \frac{2z}{z-1} + z \to \frac{z^2+z}{z-1} = \frac{z(z+1)}{z-1}.$$

$$\therefore C(z) = \frac{z(z+1)}{(z-1)(z^2-1.5z+1)}.$$

$$\frac{C(z)}{z} = \frac{z+1}{(z-1)(z^2-1.5z+1)}.$$

$$= \frac{A}{z-1} + \frac{Bz+C}{z^2-1.5z+1}$$

$$= \frac{4z}{z-1} + \frac{-4z}{z^2-1.5z+1}$$

$$=$$

$$A = \left.\frac{z+1}{z^2-1.5z+1}\right|_{z=1}$$

$$= \frac{2}{1-1.5+1} = \frac{2}{0.5}$$

$$\boxed{A = ④}$$

$$A(z^2-1.5z+1) + (Bz+C)(z-1).$$

$$\therefore A+B=0.$$

$$B=-4.$$

$$A-C=1.$$

$$C = A-1 \Rightarrow 3$$

$$\frac{C(z)}{z} = \frac{A}{z-1} + \frac{k_1}{z-0.75-j0.66} + \frac{k_1^*}{z-0.75+j0.66}$$

$$C(z) = \frac{Az}{z-1} + \frac{k_1 z}{z-0.75-j0.66} + \frac{k_1^* z}{z-0.75+j0.66}$$

$$A = \left.\frac{z+1}{z^2-1.5z+1}\right|_{z=1} = 4$$

$$k_1 = \left.\frac{z+1}{(z-1)(z-0.75+j0.66)}\right|_{z=0.75+j0.66} = \frac{0.75+j0.66+1}{(0.75+j0.66-1)(2j0.66)}$$

$$k_1 = \frac{1.75+j0.66}{-0.33j-0.8712} = -2 + j2.289\times10^{-3} = 2\angle 180.$$

$$k_1^* = -2 - j2.889\times10^{-3} = 2\angle -180.$$

$$\therefore P_1 = 0.75 \pm j0.66 = 1\angle 41.3.$$

$$aT = \ln|P_1| \Rightarrow aT = \ln 1 \Rightarrow a = 0$$

$$bT = (41.3) = \frac{\times\pi}{180} = \pi/2.$$

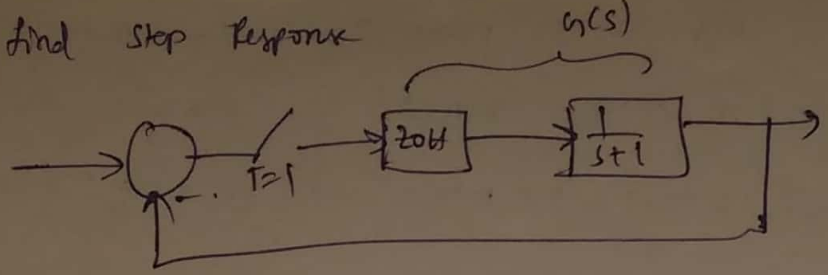
41?

$$A = 2|k_1| = 2(2) = 4$$

$$\theta = 180.$$

$$\therefore c(k) = 4(1)^k + \cos(\pi/2\, k + \pi).$$

Pr-2 Find Step Response



Sol

$$C(z) = \frac{R(z)\;G(z)}{1+G(z)}\;, \qquad R(z) = \frac{z}{z-1}$$

$$G(z) = z.T\left[\frac{1-e^{-sT}}{s}\cdot\frac{1}{s+1}\right] = \left(1-z^{-1}\right)\; z.T\left[\frac{1}{s(s+1)}\right]$$

$$= \left(\frac{z-1}{z}\right) z.T\left[\frac{1}{s} - \frac{1}{s+1}\right].$$

$$= \left[\frac{z-1}{z}\right]\; z.T\left[u(t) - e^{-t}u(t)\right]$$

$$= \frac{z-1}{z}\cdot\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] \qquad T=1.$$

$$G(z) = 1 - \frac{z-1}{z-e^{-1}} = \frac{\cancel{z}-e^{-1}-\cancel{z}+1}{z-e^{-1}} = \frac{0.63}{z-0.368}$$

$$\therefore\; C(z) = \frac{\frac{z}{z-1}\cdot\frac{0.62}{z-e^{-1}}}{1+\frac{0.62}{z-e^{-1}}} = \frac{\frac{z(0.62)}{(z-1)\cancel{(z-e^{-1})}}}{\frac{z-e^{-1}+0.62}{\cancel{(z-e^{-1})}}}$$

$$C(z) = \frac{z(0.63)}{(z-1)(z-2e^{-1}+1)}.$$

$$\frac{C(z)}{z} = \frac{0.63}{(z-1)(z-0.264)} = \frac{A}{z-1} + \frac{B}{z-0.264}.$$

$$A = \left.\frac{0.63}{z-0.264}\right|_{z=1} = \frac{0.63}{1-0.264} = 0.85\;; \qquad B = \left.\frac{0.63}{z-1}\right|_{z=0.264} = -0.85.$$

$$\therefore\; C(z) = 0.85\,\frac{z}{z-1} - 0.85\,\frac{z}{z-0.264}$$

$$c(k) = 0.85\left[1^{k} - (0.264)^{k}\right].$$